Lecture 18

Consequences of the Lebesgue–Radon–Nikodym theorem
Duality for L^p spaces

Graduate Real Analysis

Fall Semester

Auxiliary Lemma

Lemma

If $z_1, \ldots, z_N \in \mathbb{C}$, then there is $S \subseteq \{1, \ldots, N\}$ such that

$$\Big|\sum_{k\in S} z_k\Big| \geq \frac{1}{\pi}\sum_{k=1}^{N} |z_k|.$$

Proof. Let $z_k = e^{i\theta_k}|z_k|$, and $S(\theta) = \{k \in \{1, \ldots, N\} : \cos(\theta_k - \theta) > 0\}$ for $-\pi \leq \theta \leq \pi$. Then

$$\Big|\sum_{k\in S(\theta)} z_k\Big| = \Big|\sum_{k\in S(\theta)} e^{-i\theta} z_k\Big| \geq \mathrm{Re}\sum_{k\in S(\theta)} e^{-i\theta} z_k = \sum_{k=1}^{N} |z_k|\cos^+(\theta_k - \theta).$$

Choose $\theta = \theta_0 \in [-\pi, \pi]$ to maximize the left sum and set $S = S(\theta_0)$, then

$$\sum_{k=1}^{N} |z_k|\cos^+(\theta_k - \theta_0) \geq \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{N} |z_k|\cos^+(\theta_k - \theta)d\theta = \frac{1}{\pi}\sum_{k=1}^{n} |z_k|,$$

because $\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^+(\theta_k - \theta)d\theta = \frac{1}{\pi}$. This proves the lemma. $\square$

Theorem

Theorem

If μ is a complex measure on $(X, \mathcal{M})$, then $|\mu|(X) < \infty$.

Proof. Suppose $|\mu|(E) = \infty$ for some $E \in \mathcal{M}$. Put $t = \pi(1 + |\mu(E)|)$ since $|\mu|(E) > t$. There is a partition $(E_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}$ of E such that

$$\sum_{i=1}^{N} |\mu(E_i)| > t \quad \text{for some} \quad N \in \mathbb{N}.$$

By the previous lemma with $z_i = \mu(E_i)$ we obtain that there is $A \subseteq E$ and

$$|\mu(A)| > \frac{t}{\pi} > 1.$$

Setting $B = E \setminus A$ it follows that

$$|\mu(B)| = |\mu(E) - \mu(A)| \geq |\mu(A)| - |\mu(E)| > \frac{t}{\pi} - |\mu(E)| = 1.$$

Proof

- We have split $E = A \cup B$, into disjoint sets, where $|\mu(A)| > 1$ and $|\mu(B)| > 1$. Then $|\mu|(A) = \infty$ or $|\mu|(B) = \infty$, since $|\mu|$ is a measure and $|\mu|(E) = \infty$.
- Since $|\mu|(X) = \infty$, split X into A_1 and B_1 as above with $|\mu(A_1)| > 1$ and $|\mu|(B_1) = \infty$. Next, we split B_1 into A_2, B_2 with $|\mu(A_2)| > 1$ and $|\mu|(B_2) = \infty$ and so on.
- We obtain a countable disjoint collection $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}$ such that

$$\mu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) = \sum_{j\in\mathbb{N}} \mu(A_j)$$

  by the countable additivity of μ.
- This implies that $\lim_{n\to\infty} \mu(A_n) = 0$, which is impossible, since $|\mu(A_j)| > 1$ for all $j \in \mathbb{N}$. Thus we must have $|\mu|(X) < \infty$. □

The Lebesgue–Radon–Nikodym theorem revised

Theorem (The Lebesgue–Radon–Nikodym theorem for complex measures)

Let ν be a complex measure (σ-finite signed measure) and μ be a σ-finite positive measure on $(X, \mathcal{M})$. Then the following are true.

(i) (**Lebesgue decomposition**) *There exist unique complex measures (σ-finite signed measure) ν_a and ν_s on $(X, \mathcal{M})$ such that*

$$\nu_a \ll \mu \quad \text{and} \quad \nu_s \perp \mu \quad \text{and} \quad \nu = \nu_a + \nu_s.$$

(ii) (**Radon–Nikodym theorem**) *Moreover, there is $f \in L^1(X, \mu)$ (there is an extended μ-measurable function $f : X \to \mathbb{R}$) such that*

$$\nu_a(E) = \int_E f(x) d\mu(x) \quad \text{for} \quad E \in \mathcal{M},$$

and any two such functions are equal μ-a.e. on X.

Lebesgue decomposition and Radon–Nikodym derivative

Definition (Lebesgue decomposition and Radon–Nikodym derivative)

- The decomposition $\nu = \lambda + \rho$, where

$$\lambda \perp \mu \quad \text{and} \quad \rho \ll \mu$$

  is called **the Lebesgue decomposition of ν with respect to μ**.
- If $\nu \ll \mu$ by the **Radon–Nikodym** theorem we obtain $d\nu = f\,d\mu$ and f is called **the Radon–Nikodym derivative of ν with respect to μ.**
- The Radon–Nikodym derivative is sometimes denoted by

$$f = \frac{d\nu}{d\mu}, \quad \text{i.e.} \quad d\nu = \frac{d\nu}{d\mu} d\mu.$$

  More precisely, $\frac{d\nu}{d\mu}$ should be constructed as the class of of functions equal to f μ-a.e.

Useful facts about Radon–Nikodym derivatives

Proposition

Suppose that ν is a σ-finite signed measure and μ, λ are σ-finite measures of $(X, \mathcal{M})$ such that $\nu \ll \mu$ and $\mu \ll \lambda$.

(a) *If $g \in L^1(X, \nu)$, then $g\frac{d\nu}{d\mu} \in L^1(X, \mu)$ and*

$$\int_X g\, d\nu = \int_X g \frac{d\nu}{d\mu}\, d\mu.$$

(b) *We have $\nu \ll \lambda$ and*

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda},$$

holds λ-a.e. on X

(c) *In particular, by (b) we may conclude the following: if $\mu \ll \lambda$ and $\lambda \ll \mu$, then $\frac{d\lambda}{d\mu}\frac{d\mu}{d\lambda} = 1$ holds a.e. on X with respect to either μ or λ.*

Proof

Proof. By considering ν^+ and ν^- we may assume that $\nu \geq 0$.

- The equation is true when $g = \mathbb{1}_E$ and $E \in \mathcal{M}$ since
$$\int_E d\nu = \int_E \frac{d\nu}{d\mu}\, d\mu$$
by the Radon–Nikodym theorem.
- It remains true for simple functions by linearity, then for nonnegative measurable functions by the **(MCT)**, and finally for function in $L^1(X,\nu)$ by linearity again.
- Note that $\nu \ll \lambda$ and
$$\int_E \frac{d\nu}{d\lambda}\, d\lambda = \nu(E) = \int_E \frac{d\nu}{d\mu}\, d\mu = \int_E \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}\, d\lambda,$$
by applying the previous part with $g = \mathbb{1}_E \frac{d\nu}{d\mu}$ for all $E \in \mathcal{M}$.
- Thus $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$ λ-a.e. on X as desired. $\square$

Theorem

Theorem (Polar decomposition)

Let μ be a complex measure on $(X, \mathcal{M})$. Then there is a measurable function $h : X \to \mathbb{C}$ such that $|h(x)| = 1$ for all $x \in X$ and

$$d\mu = f\, d|\mu|.$$

Proof. Since $\mu \ll |\mu|$ so by the Radon–Nikodym theorem there is $h \in L^1(X, |\mu|)$ such that $d\mu = h\, d|\mu|$. Let

$$A_r = \{x \in X : |h(x)| < r\} \quad \text{for} \quad r > 0.$$

Let $(E_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}$ be such that $A_r = \bigcup_{j\in\mathbb{N}} E_j$, then

$$\sum_{j\in\mathbb{N}} |\mu(E_j)| = \sum_{j\in\mathbb{N}} \left| \int_{E_j} h\, d|\mu| \right| \leq \sum_{j\in\mathbb{N}} r|\mu|(E_j) = r|\mu|(A_r)$$

and $|\mu|(A_r) \leq r|\mu|(A_r)$, giving $|\mu|(A_r) = 0$ if $r < 1$. Thus $|h| \geq 1$ μ-a.e.

Proof

If $|\mu|(E) > 0$, by using $d\mu = h\,d|\mu|$, we obtain

$$\left|\frac{1}{|\mu|(E)}\int_E h\,d|\mu|\right| = \frac{|\mu(E)|}{|\mu|(E)} \leq 1.$$

Proposition (See previous lecture)

Suppose that $(X, \mathcal{M}, \mu)$ is a finite measure space $\mu(X) < \infty$. Let $f \in L^1(X)$ and $S \subseteq \mathbb{C}$ be a closed set and

$$A_E f = \frac{1}{\mu(E)}\int_E f d\mu \in S$$

for every $E \in \mathcal{M}$ with $\mu(E) > 0$. Then $f(x) \in S$ for almost all $x \in X$.

By this proposition with the closed unit disc in place of S, we conclude that $|h| \leq 1$ μ-a.e. Hence $|\mu|(B) = 0$, where $B = \{x \in X : |h(x)| \neq 1\}$. If we redefine h on B so that $h(x) = 1$ on B we obtain the function with the desired properties. □

Integral representation of the total variation

Theorem

Suppose μ is a positive measure on $(X, \mathcal{M})$, and $g \in L^1(X, \mu)$ and

$$\lambda(E) = \int_E g \, d\mu \quad \text{for} \quad E \in \mathcal{M}.$$

Then $|\lambda|(E) = \int_E |g| \, d\mu$.

Proof. By the previous theorem there is $h : X \to \mathbb{C}$, such that $|h(x)| = 1$ and $d\lambda = h \, d|\lambda|$, thus

$$g \, d\mu = d\lambda = hd\,|\lambda|$$

this gives

$$d|\lambda| = \overline{h}g \, d\mu.$$

Since $|\lambda| \geq 0$ and $\mu \geq 0$ it follows that $\overline{h}g \geq 0$ μ-a.e. so that $\overline{h}g = |g|$, μ-a.e., and this completes the proof. $\square$

Integral representation of the total variation

Corollary

If ν is a complex measure on $(X, \mathcal{M})$ then

$$\left| \int_X f \, d\nu \right| \leq \int_X |f| \, d|\nu| \quad \text{for any} \quad f \in L^1(X, \nu).$$

Moreover, $|\nu|(E) = \alpha(E)$, where

$$\alpha(E) = \sup \left\{ \left| \int_E f \, d\nu \right| : |f| \leq 1 \right\} \quad \text{for any} \quad E \in \mathcal{M}.$$

Proof. By the polar decomposition $d\nu = hd|\nu|$, where $|h| = 1$, so

$$\left| \int_X f \, d\nu \right| = \left| \int_X hf \, d|\nu| \right| \leq \int_X |f| \, d|\nu|.$$

This shows $\alpha(E) \leq |\nu|(E)$. We also have $\alpha(E) \geq \left| \int_E \overline{h} \, d\nu \right| = |\nu|(E)$. $\square$

The Hahn decomposition theorem

Theorem (The Hahn decomposition theorem)

Let ν be a signed measure on $(X, \mathcal{M})$, then there exist $P, N \in \mathcal{M}$ such that $P \cup N = X$ and $P \cap N = \emptyset$ and

$$\nu^+(E) = \nu(P \cap E) \quad \text{and} \quad \nu^-(E) = -\nu(N \cap E) \quad \text{for} \quad E \in \mathcal{M}.$$

If P', N' is another such pair, then $P \triangle P' = N \triangle N'$ is null for ν.

Proof. By the polar decomposition $d\nu = hd|\nu|$, where $|h| = 1$. But ν is real, it follows that h is real (a.e., and therefore everywhere, by redefining on a set of measure 0), hence $h(x) = \pm 1$ for all $x \in X$. Let

$$P = \{x \in X : h(x) = 1\} \quad \text{and} \quad N = \{x \in X : h(x) = -1\}.$$

Then $P \cup N = X$ and $P \cap N = \emptyset$.

Proof

- Since $\nu^+ = \frac{1}{2}(|\nu| + \nu)$ and

$$\frac{1}{2}(h(x)+1) = \begin{cases} h(x) & \text{if } x \in P, \\ 0 & \text{if } x \in N, \end{cases}$$

  we obtain that

$$\nu^+(E) = \frac{1}{2}\int_E (1+h)d|\nu| = \int_{E\cap P} hd|\nu| = \nu(P \cap E).$$

- Since $\nu(E) = \nu(P \cap E) + \nu(N \cap E)$ and $\nu = \nu^+ - \nu^-$ we obtain that

$$\nu^+(E) = \nu^+(E) - \nu(E) = -\nu(N \cap E).$$

- If P', N' is another pair of sets in the statement of theorem we have $P \setminus P' \subseteq P$ and $P \setminus P' \subseteq N'$, so $P \setminus P'$ is both positive and negative hence null, likewise $P' \setminus P$. Then $P \triangle P' = N \triangle N'$ is null for ν. $\square$

Jordan decomposition theorem

Theorem (Jordan decomposition theorem)

Let ν be a signed measure on $(X, \mathcal{M})$, then there exist unique positive measures ν^+ and ν^- such that

$$\nu = \nu^+ - \nu^- \quad \text{and} \quad \nu^+ \perp \nu^-.$$

Moreover, ν^+, ν^- are minimal in the sense that if $\nu = \mu_1 - \mu_2$ for two positive measures μ_1, μ_2 on $(X, \mathcal{M})$, then $\mu_1 \geq \nu^+$ and $\mu_2 \geq \nu^-$.

Proof. Let $X = P \cup N$ be the Hahn decomposition for ν, then

$$\nu^+(E) = \nu(P \cap E) \quad \text{and} \quad \nu^-(E) = -\nu(N \cap E) \quad \text{for} \quad E \in \mathcal{M}.$$

Clearly $\nu^+ \perp \nu^-$, since $P \cap N = \emptyset$. Since $\nu \leq \mu_1$ and $\nu \geq -\mu_2$ we obtain

$$\nu^+(E) = \nu(P \cap E) \leq \mu_1(P \cap E) \leq \mu_1(E)$$
$$-\nu^-(E) = \nu(N \cap E) \geq -\mu_2(N \cap E) \geq -\mu_2(E).$$

Proof

Finally we show uniqueness in the Jordan decomposition.

- If $\nu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$ let $E, F \in \mathcal{M}$ be such that $E \cap F = \emptyset$, $E \cup F = X$ and
$$\mu^+(F) = \mu^-(E) = 0.$$
- Then $X = E \cup F$ is another Hahn decomposition for ν so $P \triangle E$ is ν-null. Thus for any $A \in \mathcal{M}$ we obtain
$$\mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P) = \nu^+(A)$$
and likewise $\nu^- = \mu^-$. □

Linear functional on L^p spaces

Let $(X, \mathcal{M}, \mu)$ be a fixed measure space. Some observations are in order.

- Suppose that p and q are conjugate exponents, i.e., $1/p + 1/q = 1$. For each $g \in L^q(X)$, we define a linear functional ϕ_g on $L^p(X)$ by
$$\phi_g(f) = \int_X fg d\mu.$$
- By Hölder's inequality, we have
$$|\phi_g(f)| = \left| \int_X fg d\mu \right| \leq \|g\|_{L^q} \|f\|_{L^p},$$
so ϕ_g is a bounded linear functional and $\|\phi_g\|_{(L^p)^*} \leq \|g\|_{L^q}$.
- Especially in the $p = 2$ case, it may be more appropriate to define $\phi_g(f) = \int_X f\overline{g} d\mu$, and the same convention can be used for $p \neq 2$ without changing the results ahead in any essential way.
- Recall that a measure μ is **semifinite** if for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ such that $F \subseteq E$ and $0 < \mu(F) < \infty$.

Isometry from L^q into $(L^p)^*$

Proposition

Let $(X, \mathcal{M}, \mu)$ be measure space. Suppose that the exponents $p, q > 0$ satisfy $1 \leq q < \infty$ and $1/p + 1/q = 1$. If $g \in L^q$, then

$$\|g\|_{L^q} = \|\phi_g\|_{(L^p)^*} = \sup\left\{\left|\int_X fgd\mu\right| : \|f\|_{L^p} = 1\right\}.$$

If μ is semifinite, this result also holds for $q = \infty$.

Proof. Hölder's inequality gives that $\|\phi_g\|_{(L^p)^*} \leq \|g\|_{L^q}$, and equality is trivial if $g = 0$ almost everywhere. Otherwise, we show that the requisite supremum is attained for a specific function f.

- If $1 < q < \infty$, let

$$f = \frac{|g|^{q-1}\overline{\mathrm{sgn} g}}{\|g\|_{L^q}^{q-1}}.$$

Proof

Then

$$\|f\|_{L^p}^p = \frac{\int_X |g|^{(q-1)p} d\mu}{\|g\|_{L^q}^{(q-1)p}} = \frac{\int_X |g|^q d\mu}{\int_X |g|^q d\mu} = 1,$$

so

$$\|\phi_g\|_{(L^p)^*} \geq \int_X fg d\mu = \frac{\int_X |g|^q d\mu}{\|g\|_{L^q}^{q-1}} = \|g\|_{L^q}.$$

- If $q = 1$, let $f = \overline{\mathrm{sgn} g}$. Then $\|f\|_{L^\infty} = 1$ and $\int_X fg d\mu = \|g\|_{L^1}$.
- If $q = \infty$, let $\varepsilon > 0$ and let

  $$A = \{x \in X : |g(x)| > \|g\|_{L^\infty} - \varepsilon\}.$$

  Then $\mu(A) > 0$, so, if μ is semifinite, there exists $B \subset A$ with $0 < \mu(B) < \infty$. Let $f = \mu(B)^{-1} \mathbb{1}_B \overline{\mathrm{sgn} g}$. Then $\|f\|_{L^1} = 1$ and

  $$\|\phi_g\|_{(L^1)^*} \geq \int_X fg d\mu = \frac{1}{\mu(B)} \int_B |g| d\mu \geq \|g\|_{L^\infty} - \varepsilon.$$

  Since $\varepsilon > 0$ was arbitrary, this shows that $\|\phi_g\|_{(L^1)^*} = \|g\|_{L^\infty}$. □

Useful result

Theorem

Let $(X, \mathcal{M}, \mu)$ be measure space. Let $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. Let g be a measurable function on X such that $fg \in L^1(X)$ for all f in the space Σ of simple functions that vanish outside of a set of finite measure. Suppose that the quantity

$$M_q(g) = \sup \left\{ \left| \int_X fg d\mu \right| : f \in \Sigma \ \ \textit{and} \ \ \|f\|_{L^p} = 1 \right\}$$

is finite and suppose also that either the set $S_g = \{x \in X : g(x) \neq 0\}$ is σ-finite or μ is semifinite. Then $g \in L^q(X)$ and $M_q(g) = \|g\|_{L^q}$.

Proof. First, we remark that, if f is a bounded measurable function that vanishes outside a set E of finite measure and $\|f\|_{L^p} = 1$, then

$$\left| \int_X fg d\mu \right| \leq M_q(g).$$

Proof

- Indeed, take a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions with $|f_n| \leq |f|$ and $f_n \xrightarrow[n\to\infty]{} f$ μ-a.e. Since $|f_n| \leq \|f\|_{L^\infty} \mathbb{1}_E$ and $\mathbb{1}_E g \in L^1(X)$, by the **(DCT)**, we have that

$$\Big| \int_X fg d\mu \Big| = \lim_{n\to\infty} \Big| \int_X f_n g d\mu \Big| \leq M_q(g).$$

- Now suppose that $q < \infty$. We may assume that S_g is σ-finite since this condition automatically holds when μ is semifinite. Show this!
- Let $(E_n)_{n\in\mathbb{N}}$ be an increasing sequence of sets of finite measure such that $S_g = \bigcup_{n\in\mathbb{N}} E_n$. Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of simple functions such that $\phi_n \xrightarrow[n\to\infty]{} g$ pointwise and $|\phi_n| \leq |g|$. Let $g_n = \phi_n \mathbb{1}_{E_n}$. Then $g_n \xrightarrow[n\to\infty]{} g$ pointwise, $|g_n| \leq |g|$, and g_n vanishes outside E_n. Let

$$f_n = \frac{|g_n|^{q-1}\overline{\operatorname{sgn} g}}{\|g_n\|_{L^q}^{q-1}}.$$

  We have $\|f_n\|_{L^p} = 1$ as in the proof of the previous proposition.

Proof

- By Fatou's lemma, we have

$$\begin{aligned}\|g\|_{L^q} &\leq \liminf_{n\to\infty} \|g_n\|_{L^q} = \liminf_{n\to\infty} \int_X |f_n g_n| d\mu \\ &\leq \liminf_{n\to\infty} \int_X |f_n g| d\mu = \liminf_{n\to\infty} \int_X f_n g d\mu \leq M_q(g).\end{aligned}$$

- On the other hand, Hölder's inequality gives $M_q(g) \leq \|g\|_{L^q}$, so the proof is complete in the $1 \leq q < \infty$ case.
- If $q = \infty$, let $\varepsilon > 0$ and define $A = \{x \in X : |g(x)| \geq M_\infty(g) + \varepsilon\}$. If $\mu(A) > 0$, we could choose $B \subset A$ with $0 < \mu(B) < \infty$ (using either that μ is semifinite if $\mu(A) = \infty$ or that $A \subset S_g$).
- Setting $f = \mu(B)^{-1} \mathbb{1}_B \overline{\mathrm{sgn} g}$, we would then have $\|f\|_{L^1} = 1$ and $\int_X fg d\mu = \mu(B)^{-1} \int_B |g| d\mu \geq M_\infty(g) + \varepsilon$. But this contradicts our remark at the beginning of the proof. Therefore, $\|g\|_{L^\infty} \leq M_\infty(g)$. The reverse inequality is obvious. $\square$

Duality for L^p spaces

Theorem (Duality for L^p spaces)

Let $(X, \mathcal{M}, \mu)$ be measure space. Let $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. If $1 < p < \infty$, then, for each $\phi \in (L^p(X))^*$, there exists $g \in L^q(X)$ such that

$$\phi(f) = \int_X fg d\mu \quad \text{for all} \quad f \in L^p(X).$$

Hence, $L^q(X)$ is isometrically isomorphic to $(L^p(X))^*$. The same conclusion holds for $p = 1$ provided that μ is σ-finite.

Proof. First, suppose that $\mu(X) < \infty$, so that all simple functions are in $L^p(X)$. If $\phi \in (L^p(X))^*$ and $E \in \mathcal{M}$, let $\nu(E) = \phi(\mathbb{1}_E)$. For any disjoint sequence $(E_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}$, if $E = \bigcup_{j\in\mathbb{N}} E_j$, we have $\mathbb{1}_E = \sum_{j\in\mathbb{N}} \mathbb{1}_{E_j}$, where the series converges in $L^p(X)$ norm:

$$\Big\|\mathbb{1}_E - \sum_{j\in\mathbb{N}} \mathbb{1}_{E_j}\Big\|_{L^p} = \Big\| \sum_{j=n+1}^{\infty} \mathbb{1}_{E_j}\Big\|_{L^p} = \mu\Big(\bigcup_{j=n+1}^{\infty} E_j\Big)^{1/p} \xrightarrow[n\to\infty]{} 0.$$

Proof

- (Note that this step is where we require that $p < \infty$). Hence, since ϕ is linear and continuous, we have

$$\nu(E) = \phi(\mathbb{1}_E) = \sum_{j\in\mathbb{N}} \phi(\mathbb{1}_{E_j}) = \sum_{j\in\mathbb{N}} \nu(E_j),$$

  so ν is a complex measure. Also, if $\mu(E) = 0$, then $\mathbb{1}_E = 0$ considered as an element of $L^p(X)$, so $\nu(E) = 0$. This shows that $\nu \ll \mu$.
- By the Radon–Nikodym theorem, there exists $g \in L^1(X)$ such that

$$\phi(\mathbb{1}_E) = \nu(E) = \int_E g d\mu \quad \text{for all} \quad E \in \mathcal{M}.$$

- Therefore, $\phi(f) = \int_X fgd\mu$ for all simple functions f. Moreover, $\left| \int_X fgd\mu \right| \le \|\phi\|_{(L^p)^*} \|f\|_{L^p}$, so $g \in L^q(X)$ by the previous theorem.
- Since simple functions are dense in $L^p(X)$, we have $\phi(f) = \int_X fgd\mu$ for all $f \in L^p(X)$.

Proof

- Now suppose that μ is σ-finite. Let $(E_n)_{n\in\mathbb{N}}$ be an increasing sequence of sets such that $0 < \mu(E_n) < \infty$ and $X = \bigcup_{n\in\mathbb{N}} E_n$.
- We identify $L^p(E_n)$ and $L^q(E_n)$ with the subspaces of $L^p(X)$ and $L^q(X)$ respectively consisting of functions that vanish outside E_n.
- The preceding argument shows that, for each n, there exists $g_n \in L^q(E_n)$ such that $\phi(f) = \int_X fg_n d\mu$ for all $f \in L^p(E_n)$ and $\|g_n\|_{L^q(E_n)} = \|\phi\|_{(L^p(E_n))^*} \leq \|\phi\|_{(L^p)^*}$.
- The function g_n is unique up to a nullset, so $g_n = g_m$ almost everywhere on E_n for $n < m$. Therefore, we can define g almost everywhere on X by setting $g = g_n$ on E_n.
- Moreover, if $f \in L^p(X)$, then, by the **(DCT)**, we have $f\mathbb{1}_{E_n} \xrightarrow[n\to\infty]{} f$ in $L^p(X)$ norm, and consequently

$$\phi(f) = \lim_{n\to\infty} \phi(f\mathbb{1}_{E_n}) = \lim_{n\to\infty} \int_{E_n} fgd\mu = \int_X fgd\mu.$$

Proof

- Finally, suppose that μ is arbitrary and $p > 1$, so $q < \infty$. For each σ-finite set $E \in \mathcal{M}$, there is a unique $g_E \in L^q(E)$ (up to a nullset) such that $\phi(f) = \int_X f g_E \, d\mu$ for all $f \in L^p(E)$ and $\|g_E\|_{L^q} \leq \|\phi\|_{(L^p)^*}$.
- If F is σ-finite and $F \supset E$, then $g_F = g_E$ μ-a.e. on E, hence we have $\|g_F\|_{L^q} \geq \|g_E\|_{L^q}$. Let M be the supremum of $\|g_E\|_{L^q}$ as E ranges over all σ-finite sets, noting that $M \leq \|\phi\|_{(L^p)^*}$. Choose a sequence $(E_n)_{n\in\mathbb{N}}$ so that $\|g_{E_n}\|_{L^q} \xrightarrow[n\to\infty]{} M$ and set $F = \bigcup_{n\in\mathbb{N}} E_n$. Then F is σ-finite and $\|g_F\|_{L^q} \geq \|g_{E_n}\|_{L^q}$ for all $n \in \mathbb{N}$. Therefore, $\|g_F\|_{L^q} = M$.
- Now, if A is a σ-finite set containing F, we have

$$\int_X |g_F|^q d\mu + \int_X |g_{A\setminus F}|^q d\mu = \int_X |g_A|^q d\mu \leq M^q = \int_X |g_F|^q d\mu.$$

- Thus, $g_{A\setminus F} = 0$ and $g_A = g_F$ almost everywhere. (Here we used that $q < \infty$). But if $f \in L^p(X)$, then $A = F \cup \{x \in X : f(x) \neq 0\}$ is σ-finite, so $\phi(f) = \int_X f g_A \, d\mu = \int_X f g_F \, d\mu$. Thus, we may take $g = g_F$, and the proof is complete. $\square$