Lecture 19

Hardy–Littlewood maximal function
Lebesgue differentiation theorem

Graduate Real Analysis

Fall Semester

Vitali's covering lemma

Let $(X, \mathcal{M}) = (\mathbb{R}^d, \mathrm{Bor}(\mathbb{R}^d))$, and let λ_d be Lebesgue measure on $\mathbb{R}^d$.

Lemma (Vitali's covering lemma)

Let $\mathcal{C}$ be a collection of open balls in $\mathbb{R}^d$ and let

$$U = \bigcup_{B \in \mathcal{C}} B.$$

If $\lambda_d(U) > c$, then there exist disjoint balls $B_1, \ldots, B_n \in \mathcal{C}$ such that

$$c < 3^d \sum_{j=1}^{n} \lambda_d(B_j).$$

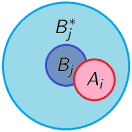
Proof. If $c < \lambda_d(U)$ then from the inner regularity of Lebesgue measure λ_d, there is a compact set $K \subseteq U$ with $\lambda_d(K) > c$ and finitely many balls $A_1, \ldots, A_m \in \mathcal{C}$ covering K, i.e. $K \subseteq \bigcup_{j=1}^{m} A_j$.

Proof

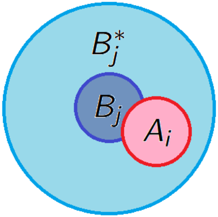
- Let B_1 be the largest of the A_j's (the one having maximal radius).
- Let B_2 be the largest of the A_j's that is disjoint from B_1.
- Let B_3 be the largest of A_j's that is disjoint from B_1 and B_2 and so on until list of A_j's is exhausted.
- According to this construction, if A_i is not one of the B_j's there is a $j \in \{1, \ldots, k\}$ such that $A_i \cap B_j \neq \emptyset$ and if j is the smallest integer with this property the radius of A_i is at most that of B_j.
- Hence

$$A_i \subseteq B_j^*,$$

where B_j^* is the ball concentric with B_j whose radius is three times that of B_j. In other words, $B_j^* = 3B_j$.



Proof



- But then

$$K \subseteq \bigcup_{j=1}^{m} A_j \subseteq \bigcup_{j=1}^{k} B_j^*,$$

and consequently

$$c < \lambda_d(K) \leq \sum_{j=1}^{k} \lambda_d(B_j^*) = \sum_{j=1}^{k} \lambda_d(3B_j) = 3^d \sum_{j=1}^{k} \lambda_d(B_j). \qquad \square$$

Locally integrable function

Definition (Locally integrable function)

A measurable function $f : \mathbb{R}^d \to \mathbb{C}$ is called **locally integrable** (with respect to Lebesgue measure) if

$$\int_K |f(x)|\,dx < \infty$$

for any bounded measurable set $K \subseteq \mathbb{R}^d$.

Notation

- We shall abbreviate $d\lambda_d(x)$ to dx.
- We denote the space of locally integrable functions by $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Hardy–Littlewood averaging operator

Definition (Hardy–Littlewood averaging operator)

If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $r > 0$ we define $M_r f(x)$ to be the average of f over the open ball $B(x,r) = \{y \in \mathbb{R}^d : |x-y| < r\}$, i.e.

$$M_r f(x) = \frac{1}{\lambda_d(B(x,r))} \int_{B(x,r)} f(y)\, dy.$$

Lemma (A)

If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then $M_r f(x)$ is jointly continuous in $(x,r) \in \mathbb{R}^d \times (0,\infty)$.

Proof. We know $\lambda_d(B(x,r)) = cr^d$, where $c = \lambda_d(B(0,1))$ (from polar coordinates) and $\lambda_d(S(x,r)) = 0$, where $S(x,r) = \{y \in \mathbb{R}^d : |x-y| = r\}$. Moreover,

$$\lim_{\substack{r\to r_0 \\ x \to x_0}} \mathbb{1}_{B(x,r)} = \mathbb{1}_{B(x_0,r_0)} \quad \text{pointwise in} \quad \mathbb{R}^d \setminus S(x_0,r_0).$$

Proof

If $r < r_0 + \frac{1}{2}$ and $|x - x_0| < \frac{1}{2}$, note that

$$\mathbb{1}_{B(x,r)}(y) \leq \mathbb{1}_{B(x_0,r+1)}(y).$$

Indeed, if $|x - y| < r$, then

$$|y - x_0| \leq |x - y| + |x - x_0| \leq r + \frac{1}{2} < r_0 + 1.$$

By the **(DCT)** it follows

$$\lim_{\substack{r \to r_0 \\ x \to x_0}} \int_{B(x,r)} f(y)\, dy = \int_{B(x_0,r_0)} f(x)\, dy.$$

Hence

$$\lim_{\substack{r \to r_0 \\ x \to x_0}} M_r f(x) = M_{r_0} f(x_0). \qquad \square$$

Hardy–Littlewood maximal function

Definition (Hardy–Littlewood maximal function)

If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we define its **Hardy–Littlewood maximal function** by

$$M_* f(x) = \sup_{r>0} M_r|f|(x) = \sup_{r>0} \frac{1}{\lambda_d(B(x,r))} \int_{B(x,r)} |f(y)|\, dy, \quad x \in \mathbb{R}^d.$$

Observation

For any $\alpha > 0$ we have

$$\{x \in \mathbb{R}^d : M_* f(x) > \alpha\} = \bigcup_{r>0} \{x \in \mathbb{R}^d : M_r|f|(x) > \alpha\}.$$

Thus by Lemma (A) we have that

$$\{x \in \mathbb{R}^d : M_* f(x) > \alpha\}$$

is open in $\mathbb{R}^d$.

Hardy–Littlewood maximal theorem

Theorem (Hardy–Littlewood maximal theorem on weak L^1)

Fix $d \in \mathbb{N}$. Let

$$M_* f(x) = \sup_{r>0} \frac{1}{\lambda_d(B(x,r))} \int_{B(x,r)} |f(y)|\,dy, \quad x \in \mathbb{R}^d, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$$

be the **Hardy–Littlewood maximal function**. *There exists a constant $C > 0$ such that for all $f \in L^1(\mathbb{R}^d)$ and for all $\alpha > 0$ we have*

$$\lambda_d(\{x \in \mathbb{R}^d : M_* f(x) > \alpha\}) \le \frac{C}{\alpha} \int_{\mathbb{R}^d} |f(y)|\,dy = \frac{C}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}.$$

This inequality is often called in the literature the weak type $(1,1)$ inequality for the Hardy–Littlewood maximal function.

Proof

Proof. Define

$$E_\alpha = \{x \in \mathbb{R}^d : M_* f(x) > \alpha\} \quad \text{for} \quad \alpha > 0,$$

this set is open according to the previous observation.

- For each $x \in E_\alpha$ we can choose $r_x > 0$ such that $M_{r_x}|f|(x) > \alpha$.
- The balls $B(x, r_x) \subseteq E_\alpha$ and $E_\alpha \subseteq \bigcup_{x \in E_\alpha} B(x, r_x)$.
- By Vitali's covering lemma if $c < \lambda_d(E_\alpha)$ there exist $x_1, \dots, x_k \in E_\alpha$ such that the balls $B_j = B(x_j, r_{x_j})$ are disjoint and

$$c < 3^d \sum_{j=1}^{k} \lambda_d(B_j).$$

- Then, since $\lambda_d(B_j) \leq \frac{1}{\alpha} \int_{B_j} |f(y)|\, dy$, we obtain

$$c < 3^d \sum_{j=1}^{k} \lambda_d(B_j) \leq \frac{3^d}{\alpha} \sum_{j=1}^{k} \int_{B_j} |f(y)|\, dy \leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)|\, dy.$$

- Letting $c \to \lambda_d(E_\alpha)$ we obtain the claim. $\square$

Layer-cake decomposition for L^p norms

L^p norms

Let $(X, \mathcal{M}, \mu)$ be measure space. Recall that for $0 < p < \infty$ we have

$$\|f\|_{L^p(X)} = \left(\int_X |f(y)|^p \, d\mu(x) \right)^{1/p}$$

and for $p = \infty$ we have

$$\|f\|_{L^\infty(X)} = \mathrm{ess.sup}_{x \in X} |f(x)|.$$

Proposition (Layer-cake decomposition)

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Then, for $f \in L^p(X)$ with $0 < p < \infty$, we have

$$\|f\|_{L^p(X)}^p = p \int_0^\infty \alpha^{p-1} \mu(\{x \in X : |f(x)| > \alpha\}) d\alpha.$$

Proof

Proof. Using the Fubini-Tonelli theorem, we have

$$
\begin{aligned}
p\int_0^\infty \alpha^{p-1}\mu(\{x\in X : |f(x)|>\alpha\})d\alpha \\
&= p\int_0^\infty \alpha^{p-1}\int_X \mathbb{1}_{\{x\in X : |f(x)|>\alpha\}}d\mu(x)d\alpha \\
&= \int_X\int_0^{|f(x)|} p\alpha^{p-1}d\alpha d\mu(x) \\
&= \int_X |f(x)|^p d\mu(x) \\
&= \|f\|_{L^p(\mathbb{R}^d)}^p.
\end{aligned}
$$

This completes the proof. □

Hardy–Littlewood maximal theorem on L^p spaces

Theorem (Hardy–Littlewood maximal theorem on strong L^p)

Fix $d \in \mathbb{N}$. Let

$$M_* f(x) = \sup_{r>0} \frac{1}{\lambda_d(B(x,r))} \int_{B(x,r)} |f(y)|\, dy, \quad x \in \mathbb{R}^d, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$$

be the **Hardy–Littlewood maximal function**. *For every $1 < p \leq \infty$, there exists a constant $C > 0$ such that for all $f \in L^p(\mathbb{R}^d)$ we have*

$$\|M_* f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}.$$

This inequality is often called in the literature the strong type (p,p) inequality for the Hardy–Littlewood maximal function or we simply say that $M_* f$ satisfies L^p bounds.

Remark

- One can check that this inequality fails, when $p = 1$. Justify this!

Proof

Proof. If $p = \infty$ there is nothing to do. Assume that $p < \infty$ and $f \geq 0$.

- Define
$$f = f\mathbb{1}_{F_\alpha} + f\mathbb{1}_{F_\alpha^c} = f_\alpha + f^\alpha,$$
where
$$F_\alpha = \{x \in \mathbb{R}^d : f(x) \leq \alpha\}.$$
- By the triangle inequality
$$M_* f \leq M_* f_\alpha + M_* f^\alpha.$$
- Thus
$$\begin{aligned}
&\lambda_d(\{x \in \mathbb{R}^d : M_* f(x) > 2\alpha\}) \\
&\quad \leq \lambda_d(\{x \in \mathbb{R}^d : M_* f_\alpha(x) + M_* f^\alpha(x) > 2\alpha\}) \\
&\quad \leq \lambda_d(\{x \in \mathbb{R}^d : M_* f_\alpha(x) > \alpha\}) \\
&\quad + \lambda_d(\{x \in \mathbb{R}^d : M_* f^\alpha(x) > \alpha\}).
\end{aligned}$$

Proof

- Note that $\lambda_d(\{x \in \mathbb{R}^d : M_* f_\alpha(x) > \alpha\}) = 0$, since $M_* f_\alpha(x) \leq \alpha$.
- Consequently, by the Layer-cake decomposition and change of variables we obtain

$$\begin{aligned}
\|M_* f\|^p_{L^p(\mathbb{R}^d)} &= p \int_0^\infty \alpha^{p-1} \lambda_d(\{x \in \mathbb{R}^d : M_* f(x) > \alpha\})\, d\alpha \\
&= 2^p p \int_0^\infty \alpha^{p-1} \lambda_d(\{x \in \mathbb{R}^d : M_* f(x) > 2\alpha\})\, d\alpha \\
&\leq 2^p p \int_0^\infty \alpha^{p-1} \lambda_d(\{x \in \mathbb{R}^d : M_* f^\alpha(x) > \alpha\})\, d\alpha
\end{aligned}$$

- Note also that $f^\alpha \in L^1(\mathbb{R}^d)$. Indeed $f^\alpha = f \mathbb{1}_{F^c_\alpha}$, and consequently

$$\|f^\alpha\|_{L^1(\mathbb{R}^d)} = \int_{F^c_\alpha} |f(x)|\, dx \leq \frac{1}{\alpha^{p-1}} \int_{\mathbb{R}^d} |f(x)|^p\, dx = \frac{1}{\alpha^{p-1}} \|f\|^p_{L^p(\mathbb{R}^d)}.$$

Proof

- We apply the weak type $(1,1)$ inequality for the Hardy–Littlewood maximal function to conclude that

$$\begin{aligned}
2^p p \int_0^\infty & \alpha^{p-1} \lambda_d(\{x \in \mathbb{R}^d : M_* f^\alpha(x) \geq 2\alpha\})\, d\alpha \\
&\leq C_p 2^p p \int_0^\infty \alpha^{p-2} \int_{\mathbb{R}^d} |f^\alpha(x)|\, dx\, d\alpha \\
&= C_p 2^p p \int_0^\infty \alpha^{p-2} \int_{F_\alpha^c} |f(x)|\, dx\, d\alpha \\
&\leq C_p 2^p p \int_{\mathbb{R}^d} |f(x)| \int_0^{|f(x)|} \alpha^{p-2}\, d\alpha\, dx \qquad \text{(by Fubini)} \\
&= \frac{C_p 2^p p}{p-1} \int_{\mathbb{R}^d} |f(x)|^{p-1} |f(x)|\, dx \\
&= \frac{C_p 2^p p}{p-1} \|f\|_{L^p}^p.
\end{aligned}$$

- This completes the proof. □

A first glimpse at the Lebesgue differentiation theorem

Theorem (A)

For $d \in \mathbb{N}$, let

$$M_r f(x) = \frac{1}{\lambda_d(B(x,r))} \int_{B(x,r)} f(y)\, dy, \quad x \in \mathbb{R}^d, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$$

be **the Hardy–Littlewood averaging operator**. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then*

$$\lim_{r\to 0} M_r f(x) = f(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^d.$$

Notation

- In the proof we will use the following notion of the superior limit:

$$\limsup_{r\to R} \phi(r) = \lim_{\varepsilon\to 0} \sup_{0<|R-r|<\varepsilon} \phi(r) = \inf_{\varepsilon>0} \sup_{0<|r-R|<\varepsilon} \phi(r).$$

- One can easily verify that

$$\limsup_{r\to R} |\phi(r) - c| = 0 \quad \Longleftrightarrow \quad \lim_{r\to R} \phi(r) = c.$$

Proof

Proof. The proof illustrates a very important technique in establishing pointwise convergence.

- Since $f \in L^1_{\mathrm{loc}}$ it suffices to show that for $N \in \mathbb{N}$ we have

$$\lim_{r \to 0} M_r f(x) = f(x) \quad \text{a.e. on} \quad B(0, N).$$

- But for $|x| \leq N$ and $r \leq 1$, the averaging operator $M_r f(x)$ depends only on the values $f(y)$ for $|y| \leq N + 1$ so by replacing f with $f \mathbb{1}_{B(0,N+1)}$ we may assume that $f \in L^1(\mathbb{R}^d)$.
- Given $\varepsilon > 0$ we can find a continuous integrable function g such that

$$\int_{\mathbb{R}^d} |g(y) - f(y)| \, dy < \varepsilon.$$

Proof

- Continuity of g implies that for every $x \in \mathbb{R}^d$ and $\delta > 0$ there exists $r > 0$ such that $|g(y) - g(x)| < \delta$ whenever $|x - y| < r$ and

$$|M_r g(x) - g(x)| = \frac{1}{\lambda_d(B(x,r))} \left| \int_{B(x,r)} (g(x) - g(y))\, dy \right| < \infty.$$

- Thus

$$\lim_{r \to 0} M_r g(x) = g(x) \quad \text{for every} \quad x \in \mathbb{R}^d.$$

- This immediately implies that

$$\begin{aligned} &\limsup_{r \to 0} |M_r f(x) - f(x)| \\ &\qquad = \limsup_{r \to 0} |M_r(f - g)(x) + (M_r g(x) - g(x)) + (g(x) - f(x))| \\ &\qquad \leq M_*(f - g)(x) + |f(x) - g(x)|. \end{aligned}$$

Proof

- Hence if $E_\alpha = \{x \in \mathbb{R}^d : \limsup_{r\to 0} |M_r f(x) - f(x)| > \alpha\}$, and
$$F_\alpha = \{x \in \mathbb{R}^d : |f(x) - g(x)| > \alpha\}$$
we have
$$E_\alpha \subseteq F_{\alpha/2} \cup \{x \in \mathbb{R}^d : M_*(f-g)(x) > \alpha/2\}.$$
- By Chebyshev's inequality
$$\lambda_d(F_\alpha) \leq \frac{2}{\alpha} \int_{F_{\alpha/2}} |f(x) - g(x)|\, dx < \frac{2\varepsilon}{\alpha}.$$
- By the weak type $(1,1)$ inequality for the Hardy–Littlewood maximal function we obtain
$$\lambda_d(E_\alpha) \leq \frac{2\varepsilon}{\alpha} + \frac{2C\varepsilon}{\alpha}.$$
- Since $\varepsilon > 0$ is arbitrary $\lambda_d(E_\alpha) = 0$ for all $\alpha > 0$. Then
$$\lim_{r\to 0} M_r f(x) = f(x) \quad \text{for all} \quad x \notin \bigcup_{n\in\mathbb{N}} E_{1/n}, \quad \text{so we are done.} \quad \square$$

Corollary

Corollary

For $d \in \mathbb{N}$, let

$$M_r f(x) = \frac{1}{\lambda_d(B(x,r))} \int_{B(x,r)} f(y)\, dy, \quad x \in \mathbb{R}^d, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$$

be **the Hardy–Littlewood averaging operator**. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then*

$$\lim_{r \to 0} \frac{1}{\lambda_d(B(x,r))} \int_{B(x,r)} (f(y) - f(x))\, dy = 0 \quad \text{a.e. on} \quad \mathbb{R}^d.$$

Definition (Lebesgue set)

Let us define **the Lebesgue set** L_f of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ by

$$L_f = \left\{ x \in \mathbb{R}^d : \lim_{r \to 0} \frac{1}{\lambda_d(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0 \right\}.$$

Lebesgue sets are measure zero sets

Theorem (B)

If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then $\lambda_d(L_f^c) = 0$.

Proof. For each $c \in \mathbb{C}$ we apply Theorem (A) to $g_c(x) = |f(x) - c|$ to conclude that except on a Lebesgue null set E_c we have

$$\lim_{r \to 0} \frac{1}{\lambda_d(B(x,r))} \int_{B(x,r)} |f(y) - c| \, dy = |f(x) - c|.$$

Let D be a countable and dense set in $\mathbb{C}$ and let $E = \bigcup_{c \in D} E_c$. Then $\lambda_d(E) = 0$. If $x \notin E$, for any $\varepsilon > 0$ we can find $c \in D$ with $|f(x) - c| < \varepsilon$ so that $|f(x) - f(y)| \leq |f(y) - c| + \varepsilon$. Hence

$$\limsup_{r \to 0} \frac{1}{\lambda_d(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dy \leq |f(x) - c| + \varepsilon < 2\varepsilon,$$

thus $x \in L_f$. Hence $L_f^c \subseteq E$ and $\lambda_d(L_f^c) \leq \lambda_d(E) = 0$. $\square$

Lebesgue differentiation theorem

Definition (Nicely shrinking sets)

A family of Borel sets $(E_r)_{r>0} \subseteq \mathbb{R}^d$ is said to **shirnk nicely** to $x \in \mathbb{R}^d$ if

(a) $E_r \subseteq B(x, r)$ for each $r > 0$,

(b) there is $\alpha > 0$ independent of $r > 0$ such that $\lambda_d(E_r) \geq \alpha \lambda_d(B(x, r))$.

Theorem (Lebesgue differentiation theorem)

Suppose $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. For every $x \in L_f$, (L_f is the Lebesgue set of f), we have

$$\lim_{r \to 0} \frac{1}{\lambda_d(E_r)} \int_{E_r} |f(y) - f(x)| = 0,$$

$$\lim_{r \to 0} \frac{1}{\lambda_d(E_r)} \int_{E_r} f(y)\, dy = f(x)$$

for every $(E_r)_{r>0}$ that shrinks nicely to $x \in \mathbb{R}^d$.

Regular measure

Proof. Since $\lambda_d(L_f^c) = 0$ we obtain

$$\begin{aligned}\frac{1}{\lambda_d(E_r)} \int_{E_r} |f(y) - f(x)|\, dy &\le \frac{1}{\lambda_d(E_r)} \int_{B(x,r)} |f(y) - f(x)|\, dy \\ &\le \frac{1}{\alpha \lambda_d(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy \xrightarrow[r\to 0]{} 0\end{aligned}$$

$\square$

Definition (Regular measures)

A Borel measure ν on $\mathbb{R}^d$ is called **regular** if

1. $\nu(K) < \infty$ for every compact $K \subseteq \mathbb{R}^d$,
2. $\nu(E) = \inf\{\nu(U) : E \subseteq U,\ \text{and } U \text{ is open}\}$ for every $E \in \mathrm{Bor}(\mathbb{R}^d)$.

Remark

- In fact, (ii) is implied by (i), and this definition of regular measures coincides with the regularity discussed at the beginning of the course.
- A signed or complex Borel measure ν is called **regular** if $|\nu|$ is regular.

Theorem

Example

If $f \in L^0_+(\mathbb{R}^d)$ then the measure $f\,d\lambda_d$ is regular iff $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Why?

Theorem (Lebesgue differentiation theorem for Borel measures)

Let ν be a regular signed or complex Borel measure on $\mathbb{R}^d$ and let $d\nu = d\mu + f\,d\lambda_d$ be the Lebesgue–Radon–Nikodym decomposition. Then for λ_d-a.e. $x \in \mathbb{R}^d$ we have

$$\lim_{r\to 0} \frac{\nu(E_r)}{\lambda_d(E_r)} = f(x), \qquad \text{and} \qquad \lim_{r\to 0} \frac{\mu(E_r)}{\lambda_d(E_r)} = 0,$$

for every family $(E_r)_{r>0} \subseteq \mathbb{R}^d$ that shrinks nicely to $x \in \mathbb{R}^d$.

Proof. It is not difficult to see that $d|\nu| = d|\mu| + |f|\,d\lambda_d$ so the regularity of ν implies the regularity of both μ and $f\,d\lambda_d$.

Proof

- Since $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ so in view of the previous theorem it suffices to prove that if μ is regular and $\mu \perp \lambda_d$, then for a.e. $x \in \mathbb{R}^d$ we have

$$\lim_{r \to 0} \frac{\mu(E_r)}{\lambda_d(E_r)} = 0.$$

- It also suffices to take $B_r = B(x, r)$ and assume μ is positive since for some $\alpha > 0$ we have

$$\left| \frac{\mu(E_r)}{\lambda_d(E_r)} \right| \leq \frac{|\mu|(E_r)}{\lambda_d(E_r)} \leq \frac{|\mu|(B(x,r))}{\alpha \lambda_d(B(r,r))}.$$

- Assuming that $\mu \geq 0$ let $A \in \mathrm{Bor}(\mathbb{R}^d)$ such that $\mu(A) = \lambda_d(A^c) = 0$ and let

$$F_k = \left\{ x \in A : \limsup_{r \to 0} \frac{\mu(B(x,r))}{\lambda_d(B(x,r))} > \frac{1}{k} \right\}.$$

- We shall show that $\lambda_d(F_k) = 0$ for all $k \in \mathbb{N}$ and we will be done.

Proof

- By regularity of μ given $\varepsilon > 0$ there is an open $U_\varepsilon \supseteq A$ such that $\mu(U_\varepsilon) < \varepsilon$.
- Each $x \in F_k$ is the center of a ball $B_x \subseteq U_\varepsilon$ and $\mu(B_x) > \lambda_d(B_x)k^{-1}$.
- By Vitali's covering argument if $V_\varepsilon = \bigcup_{x \in F_k} B_x$ and $c < \lambda_d(V_\varepsilon)$ there are $x_1, \ldots, x_J$ such that $B_1, \ldots, B_J$ are disjoint and

$$\begin{aligned} c < 3^d \sum_{j=1}^{J} \lambda_d(B_{x_j}) &\leq 3^d k \sum_{j=1}^{J} \mu(B_{x_j}) \leq 3^d k \mu(V_\varepsilon) \\ &\leq 3^d k \mu(U_\varepsilon) \leq 3^d k \varepsilon. \end{aligned}$$

- We conclude that $\lambda_d(V_\varepsilon) \leq 3^d k \varepsilon$ and since $F_k \subseteq V_\varepsilon$ and $\varepsilon > 0$ is arbitrary then $\lambda_d(F_k) = 0$. $\square$