Lecture 2

Algebras, σ-algebras, semi-algebras and Borel sets

Graduate Real Analysis

Fall Semester

Algebras

Let X be a nonempty set, and let $\mathcal{P}(X)$ be its power set.

Algebra

An **algebra** of sets on X is a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ that satisfies

1. $\mathcal{A} \neq \emptyset$.
2. If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$ (closed under complements).
3. If $E_1, E_2, \ldots, E_N \in \mathcal{A}$, then $\bigcup_{j=1}^{N} E_j \in \mathcal{A}$ (closed under finite unions).

Example

$\mathcal{A} = \{\emptyset, X, E, E^c\}$ is an algebra.

σ-Algebras

σ-Algebra

A σ-**algebra** of sets on X is a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ that satisfies

1. $\mathcal{A} \neq \emptyset$.
2. If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$, (closed under complements).
3. If $E_1, E_2, \ldots \in \mathcal{A}$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$, (closed under infinite unions).

The pair $(X, \mathcal{A})$ is then called a measurable space.

Examples

- If X is any set then $\{\emptyset, X\}$ and $\mathcal{P}(X)$ are σ-algebras.
- If X is uncountable, then

$$\mathcal{A} = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}$$

is σ-**algebra of countable and co-countable sets**.

Remarks

- Algebras (resp. σ-algebras) are also closed under finite (resp. countable) intersections, since $\left(\bigcup_{j=1}^{\infty} E_j\right)^c = \bigcap_{j=1}^{\infty} E_j^c$.
- If $\mathcal{A}$ is an algebra, then $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$. Take $E \in \mathcal{A} \neq \emptyset$, then $E^c \in \mathcal{A}$, and consequently $\emptyset = E \cap E^c \in \mathcal{A}$ and $X = E \cup E^c \in \mathcal{A}$.
- An algebra $\mathcal{A}$ is a σ-algebra provided it is closed under countable disjoint unions. Indeed, let $F_1 = E_1$ and

$$F_k = E_k \setminus \left(\bigcup_{j=1}^{k-1} E_j\right) = E_k \cap \left(\bigcup_{j=1}^{k-1} E_j\right)^c \quad \text{for any} \quad k \in \mathbb{N}.$$

  Then the F_k's belong to $\mathcal{A}$ and are disjoint, and

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} F_j.$$

- The intersection of any family of σ-algebras on X is again a σ-algebra.

σ-algebras generated by $\mathcal{E}$

σ-algebra generated by $\mathcal{E}$

If $\mathcal{E} \subseteq \mathcal{P}(X)$, then

$$\sigma(\mathcal{E}) = \bigcap_{\substack{\mathcal{A} \supseteq \mathcal{E} \\ \mathcal{A} \text{ is a } \sigma\text{-algebra}}} \mathcal{A}$$

is the smallest σ-algebra containing $\mathcal{E}$.

- $\sigma(\mathcal{E})$ is called the σ-algebra generated by $\mathcal{E}$ and is unique.

Example

The family $\mathcal{E} = \{\{x\} : x \in X\}$ generates

$$\sigma(\mathcal{E}) = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}.$$

Lemma

Lemma

If $\mathcal{E} \subseteq \sigma(\mathcal{F})$, then $\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{F})$.

Proof.

We know that

$$\sigma(\mathcal{E}) = \bigcap_{\substack{\mathcal{A} \supseteq \mathcal{E} \\ \mathcal{A} \text{ is a } \sigma\text{-algebra}}} \mathcal{A} \subseteq \sigma(\mathcal{F})$$

$\square$

It is also easy to see that $\sigma(\mathcal{E}) = \sigma(\sigma(\mathcal{E}))$. Indeed,

$$\sigma(\sigma(\mathcal{E})) = \bigcap_{\substack{\mathcal{A} \supseteq \sigma(\mathcal{E}) \\ \mathcal{A} \text{ is a } \sigma\text{-algebra}}} \mathcal{A} \subseteq \sigma(\mathcal{E}) \subseteq \sigma(\sigma(\mathcal{E})).$$

Borel sets and F_σ and G_δ sets

Borel sets

If X is any metric space, or more generally any topological space, the σ-algebra generated by the family of open sets in X (or, equivalently, by the family of closed sets in X) is called the **Borel σ-algebra** on X and is denoted by

$$\mathrm{Bor}(X) \subseteq \mathcal{P}(X).$$

Its members are called **Borel sets**.

F_σ and G_δ sets

1. A countable intersection of open sets is called a G_δ set.
2. A countable union of closed sets is called an F_σ set.
3. A countable union of G_δ set is called a $G_{\delta\sigma}$ set.
4. A countable intersection of F_σ sets is called an $F_{\sigma\delta}$ sets.

$\mathrm{Bor}(\mathbb{R})$ and various generating sets

Problem 1

Prove that $\mathrm{Bor}(\mathbb{R})$ is generated by each of the following:

1. the open intervals: $\mathcal{E}_1 = \{(a, b) : a < b\}$;
2. the closed intervals: $\mathcal{E}_2 = \{[a, b] : a < b\}$;
3. the half-open intervals:
$$\mathcal{E}_3 = \{(a, b] : a < b\} \quad \text{or} \quad \mathcal{E}_4 = \{[a, b) : a < b\};$$
4. the open rays:
$$\mathcal{E}_5 = \{(a, \infty) : a \in \mathbb{R}\} \quad \text{or} \quad \mathcal{E}_6 = \{(-\infty, a) : a \in \mathbb{R}\};$$
5. the closed rays:
$$\mathcal{E}_7 = \{[a, \infty) : a \in \mathbb{R}\} \quad \text{or} \quad \mathcal{E}_8 = \{(-\infty, a] : a \in \mathbb{R}\}.$$

Product σ-algebra

Definition of the product σ-algebra

Let $(X_\alpha)_{\alpha\in A}$ be an indexed collection of nonempty sets, $X = \prod_{\alpha\in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ the coordinate maps. If $\mathcal{A}_\alpha$ is a σ-algebra on X_α for each $\alpha \in A$, **the product σ-algebra on X** is the σ-algebra generated by

$$\{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{A}_\alpha,\ \alpha \in A\}.$$

We denote this σ-algebra by $\bigotimes_{\alpha\in A} \mathcal{A}_\alpha$. In other words,

$$\bigotimes_{\alpha\in A} \mathcal{A}_\alpha = \sigma\left(\{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{A}_\alpha,\ \alpha \in A\}\right).$$

If $A = \{1, \dots, n\}$, we write $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \dots \otimes \mathcal{A}_n$.

Proposition

Proposition

If A is countable, then $\bigotimes_{\alpha \in A} \mathcal{A}_\alpha$ is the σ-algebra generated by

$$\Big\{ \prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{A}_\alpha \Big\}.$$

Proof. Let

$$\mathcal{E} = \{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{A}_\alpha,\ \alpha \in A\},$$

and

$$\mathcal{F} = \Big\{ \prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{A}_\alpha \Big\}.$$

By the definition $\sigma(\mathcal{E}) = \bigotimes_{\alpha \in A} \mathcal{A}_\alpha$.

- We first show that $\mathcal{E} \subseteq \mathcal{F} \subseteq \sigma(\mathcal{F})$, this will imply $\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{F})$.

Proof

Indeed, if $\pi_\alpha^{-1}[E_\alpha] \in \mathcal{E}$ for some $\alpha \in A$ and $E_\alpha \in \mathcal{A}_\alpha$, then

$$\pi_\alpha^{-1}[E_\alpha] = \prod_{\beta \in A} E_\beta,$$

where $E_\beta = X_\beta$ for $\alpha \neq \beta$. Thus $\pi_\alpha^{-1}[E_\alpha] \in \mathcal{F}$ as desired.

- We now show that $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{E})$. Is suffices to prove that $\mathcal{F} \subseteq \sigma(\mathcal{E})$. Note that

$$\mathcal{F} \ni \prod_{\alpha \in A} E_\alpha = \bigcap_{\alpha \in A} \pi_\alpha^{-1}[E_\alpha] \in \sigma(\mathcal{E}).$$

The fact that A is countable, is essential in the argument. □

Proposition

Proposition

Suppose that $\mathcal{A}_\alpha = \sigma(\mathcal{E}_\alpha)$ for any $\alpha \in A$. Then $\bigotimes_{\alpha\in A} \mathcal{A}_\alpha$ is generated by

$$\mathcal{F}_1 = \{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{E}_\alpha,\ \alpha \in A\}.$$

If A is countable and $X_\alpha \in \mathcal{E}_\alpha$ for any $\alpha \in A$, then $\bigotimes_{\alpha\in A} \mathcal{A}_\alpha$ is generated by

$$\mathcal{F}_2 = \Big\{ \prod_{\alpha\in A} E_\alpha \ : \ E_\alpha \in \mathcal{E}_\alpha \Big\}.$$

Proof. We have

$$\sigma(\mathcal{F}_1) \subseteq \sigma\left(\{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{A}_\alpha,\ \alpha \in A\}\right) = \bigotimes_{\alpha\in A} \mathcal{A}_\alpha.$$

We have to prove that $\bigotimes_{\alpha\in A} \mathcal{A}_\alpha \subseteq \sigma(\mathcal{F}_1)$.

Proof

- For each $\alpha \in A$, consider the collection

$$\mathcal{G}_\alpha = \{E \subseteq X_\alpha : \pi_\alpha^{-1}[E] \in \sigma(\mathcal{F}_1)\}.$$

- By the definition of $\mathcal{F}_1$ we have $\mathcal{E}_\alpha \subseteq \mathcal{G}_\alpha$, since

$$\pi_\alpha^{-1}[E] \in \sigma(\mathcal{F}_1)$$

  for all $E \in \mathcal{E}_\alpha$ and $\alpha \in A$.
- One can easily show that G_α is a σ-algebra on X_α. Why?
- Hence $\mathcal{G}_\alpha$ is a σ-algebra containing $\mathcal{E}_\alpha$ thus $\mathcal{A}_\alpha = \sigma(\mathcal{E}_\alpha) \subseteq \mathcal{G}_\alpha$ for all $\alpha \in A$, so $\{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{A}_\alpha,\ \alpha \in A\} \subseteq \sigma(\mathcal{F}_1)$ and consequently

$$\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma\left(\{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{A}_\alpha,\ \alpha \in A\}\right) \subseteq \sigma(\mathcal{F}_1)$$

  as claimed. $\square$

Borel sets on the Cartesian products

Problem 2

Let $N \in \mathbb{N}$ and $X_1, \dots, X_N$ be metric spaces and let $X = \prod_{j=1}^{N} X_j$ be equipped with the product metric. Then show that

$$\bigotimes_{j=1}^{N} \mathrm{Bor}(X_j) \subseteq \mathrm{Bor}(X).$$

If the X_j's are separable, then prove that

$$\bigotimes_{j=1}^{N} \mathrm{Bor}(X_j) = \mathrm{Bor}(X).$$

What can be said if $N = \infty$?

Semi-algebras

Semi-algebras

A **semi-algebra** of sets on X is a collection $\mathcal{I} \subseteq \mathcal{P}(X)$ such that

1. $\emptyset \in \mathcal{I}$,
2. if $E, F \in \mathcal{I}$, then $E \cap F \in \mathcal{I}$,
3. if $E \in \mathcal{I}$, then E^c is a finite disjoint union of members of $\mathcal{I}$.

Examples

- A family containing $\emptyset$, $\mathbb{R}$ and all closed-open intervals $[a, b)$ with $-\infty \leq a < b \leq \infty$ is a semi-algebra.
- A family containing $\emptyset$, $\mathbb{R}$ and all open-closed intervals $(a, b]$ with $-\infty \leq a < b \leq \infty$ is a semi-algebra.
- If we consider two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ then the set $\mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$ is a semi-algebra.

Proposition

Proposition

If $\mathcal{I}$ is a semi-algebra, the collection $\mathcal{A}$ of finite disjoint unions of members of $\mathcal{I}$ is an algebra.

Proof. If $A, B \in \mathcal{A}$ we first show that $A \cup B \in \mathcal{A}$. Note that

$$B^c = \bigcup_{j=1}^{J} C_j,$$

where $C_j \in \mathcal{I}$ are disjoint. Then

$$A \setminus B = \bigcup_{j=1}^{J} A \cap C_j \in \mathcal{A},$$

thus $A \cup B = (A \setminus B) \cup B \in \mathcal{A}$.

Proof

- By induction one can easily show that if $A_1, A_2, \ldots, A_n \in \mathcal{A}$, then

$$\bigcup_{j=1}^{n} A_j \in \mathcal{A}.$$

- We now show that if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$. If $A \in \mathcal{A}$, then it can be represented as a disjoint union of the elements from $\mathcal{I}$, i.e.

$$A = A_1 \cup A_2 \cup \ldots \cup A_n,$$

where $A_1, A_2, \ldots, A_n \in \mathcal{I}$ and

$$A_m^c = \bigcup_{j=1}^{J_m} B_m^j$$

for every $m \in \{1, 2, \ldots, n\}$ with disjoint members $B_m^1, \ldots, B_m^{J_m}$ of $\mathcal{I}$.

Proof

Finally, one sees that

$$\begin{aligned} A^c &= \left(\bigcup_{m=1}^{n} A_m \right)^c \\ &= \bigcap_{m=1}^{n} A_m^c \\ &= \bigcap_{m=1}^{n} \bigcup_{j=1}^{J_m} B_m^j \\ &= \bigcup_{j_1=1}^{J_1} \bigcup_{j_2=1}^{J_2} \cdots \bigcup_{j_n=1}^{J_n} \left(B_1^{j_1} \cap B_2^{j_2} \cap \ldots \cap B_n^{j_n} \right), \end{aligned}$$

where the last sum is disjoint. Thus $\mathcal{A}$ is an algebra as claimed. $\square$

Intervals

(a) Let J_o be the collection of all **open** intervals in $\mathbb{R}$:

$$J_o = \{(a, b) : -\infty \leq a < b \leq \infty\} \cup \{\emptyset\}.$$

(b) Let J_{co} be the collection of all **closed-open** intervals in $\mathbb{R}$:

$$J_{co} = \{[a, b) : -\infty \leq a < b \leq \infty\} \cup \{\emptyset\}.$$

(c) Let J_{oc} be the collection of all **open-closed** intervals in $\mathbb{R}$:

$$J_{oc} = \{(a, b] : -\infty \leq a < b \leq \infty\} \cup \{\emptyset\}.$$

(d) Let J_c be the collection of all **closed** intervals in $\mathbb{R}$:

$$J_c = \{[a, b] : -\infty \leq a < b \leq \infty\} \cup \{\emptyset\}.$$

(e) Let $J = J_o \cup J_{oc} \cup J_{co} \cup J_c$ be the collection of **all intervals** in $\mathbb{R}$.

Rectangles

Finally, we define the family of **all rectangles** by

$$J^d = J_o^d \cup J_{oc}^d \cup J_{co}^d \cup J_c^d,$$

where

$$J_\ell^d = \{E_1 \times \ldots \times E_d : E_1, \ldots, E_d \in J_\ell\} \quad \text{for} \quad \ell \in \{o, oc, co, c\}.$$

- In other words, $J_o^d, J_{oc}^d, J_{co}^d, J_c^d$, are, respectively, the families of **open, open-closed, closed-open,** and **closed** rectangles in $\mathbb{R}^d$.

Problem 3

Show that both J_{oc}^d and J_{co}^d are semi-algebras.

Preimages and σ-algebras

Problem 4

Let X and Y be sets and let $f : X \to Y$ be a function.

1. Let Σ be a σ-algebra of subsets of X. Show that
$$\{F \subseteq Y : f^{-1}[F] \in \Sigma\}$$
is a σ-algebra of subsets of Y.
2. Let Σ be a σ-algebra of subsets of Y. Show that $\{f^{-1}[F] : F \in \Sigma\}$ is a σ-algebra of subsets of X.

Problem 5

Let X be a set and $\mathcal{E} \subseteq \mathcal{P}(X)$. Suppose that Y is another set and $f : Y \to X$ a function. Show that $\{f^{-1}[E] : E \in \sigma(\mathcal{E})\}$ is the σ-algebra of subsets of Y generated by $\{f^{-1}[E] : E \in \mathcal{E}\}$.

$\mathrm{Bor}(X)$ for a metric space X

Problem 6

Show that the Borel σ-algebra $\mathrm{Bor}(X)$ of a metric space X equals the smallest family $\mathcal{B} \subseteq \mathcal{P}(X)$ that contains

1. all open subsets of X and is closed under countable intersections and countable unions;
2. all closed subsets of X and is closed under countable intersections and countable unions;
3. all open subsets of X and is closed under countable intersections and countable disjoint unions.

Monotone classes

Monotone class

A collection $\mathcal{M} \subseteq \mathcal{P}(X)$ is called a **monotone class** if

1. whenever $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ and $E_1 \supseteq E_2 \supseteq \ldots$, then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{M}$, (closed under nonincreasing intersections)
2. whenever $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \ldots$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$, (closed under nondecreasing unions).

Problem 7

Prove that the intersection of all monotone classes containing $\mathcal{E} \subseteq \mathcal{P}(X)$ is a unique smallest monotone class containing $\mathcal{E}$.

Problem 8

Show that if $\mathcal{A}$ is an algebra on X, then the smallest monotone class containing $\mathcal{A}$ equals $\sigma(\mathcal{A})$.

σ-algebras generated by finite families

Let $\mathcal{E} = \{E_1, \ldots, E_n\}$ be a finite collection of distinct, but not necessarily disjoint, subsets of X. Let $\mathcal{D}$ be the collection of all subsets of X of the form

$$F_1 \cap \ldots \cap F_n, \quad \text{where} \quad F_i \in \{E_i, E_i^c\} \text{ for each } i \in \{1, \ldots, n\},$$

and let $\mathcal{U}$ be the collection of all finite unions of members of $\mathcal{D}$, where the empty union is understood to be $\emptyset$.

Problem 9

1. Show that $\mathcal{D}$ has at most 2^n distinct members.
2. Show that $\sigma(\mathcal{E}) = \mathcal{U}$.
3. Show that $\mathcal{U}$ has at most 2^{2^n} distinct members.

σ-algebras and their properties

Problem 10

Let $\mathcal{M}$ be an infinite σ-algebra. Show that

1. $\mathcal{M}$ contains an infinite sequence of nonempty pairwise disjoint sets;
2. $\operatorname{card}(\mathcal{M}) \geq \mathfrak{c}$.

Problem 11

Let $\mathcal{A}$ be a σ-algebra on X and suppose that a set $S \subseteq X$ does not belong to $\mathcal{A}$. Show that $\sigma(\mathcal{A} \cup \{S\}) = \{(A \cap S) \cup (B \cap S^c) : A, B \in \mathcal{A}\}$.

Problem 12

Let $\mathcal{E} \subseteq \mathcal{P}(X)$ and let $\mathcal{M} = \sigma(\mathcal{E})$. Show that

$$\mathcal{M} = \bigcup_{\substack{\mathcal{F} \subseteq \mathcal{E} \\ \mathcal{F} \text{ is countable}}} \sigma(\mathcal{F}).$$

(Hint: show that the right-hand side is a σ-algebra.)

π-systems and λ-systems

Let X be a nonempty set.

π-system

A collection $\Pi \subseteq \mathcal{P}(X)$ is a π-**system** of subsets of X if it satisfies

1. $\Pi \neq \emptyset$,
2. if $A, B \in \Pi$, then $A \cap B \in \Pi$.

λ-system

A collection $\Lambda \subseteq \mathcal{P}(X)$ is a λ-**system** of subsets of X if it satisfies

1. $X \in \Lambda$,
2. if $A, B \in \Lambda$ and $A \subseteq B$, then $B \setminus A \in \Lambda$,
3. if $(A_n)_{n\in\mathbb{N}} \subseteq \Lambda$ and $A_1 \subseteq A_2 \subseteq \ldots$, then $\bigcup_{n=1}^{\infty} A_n \in \Lambda$.

Problem 13

Show that if $\mathcal{D}$ is a λ-system containing a π-system $\mathcal{A}$, then $\sigma(\mathcal{A}) \subseteq \mathcal{D}$.