Lecture 20

Differentiability of functions
Functions of Bounded Variation
Fundamental theorem of calculus

Graduate Real Analysis

Fall Semester

Motivations

Motivations

- We want to find a broad condition on functions $F : [a, b] \to \mathbb{R}$ that guarantees the identity

$$\int_a^b F'(x)dx = F(b) = F(a) \qquad (*).$$

There are two phenomena that make this question problematic.

- First, the identity (*) might not be meaningful if we merely assumed F was continuous due to the existence of non-differentiable functions.
- Second, even if $F'(x)$ existed for every x, the function $F'(x)$ would not necessarily be Lebesgue integrable. Consider the function

$$F(x) = \begin{cases} x^2 \sin(x^{-2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

One can show that $F'(x)$ exists for $x \in [-1, 1]$, but $F' \notin L^1([-1, 1])$.

Discontinuities of monotinic functions

Theorem

Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a monotonic function. Then the set of points of I of which f is discontinuous is at most countable.

Proof. Wlog we may assume that f is increasing and $I = \mathbb{R}$.

- Let E be the set of points at which f is discontinuous.
- With every point $x \in E$ we associate a rational number $r(x) \in \mathbb{Q}$ such that
$$f(x-) < r(x) < f(x+),$$
so $r : E \to \mathbb{Q}$. Here $f(x+) = \lim_{y \searrow x} f(y)$ and $f(x+) = \lim_{y \nearrow x} f(y)$.
- Since $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$ we see that $r(x_1) \neq r(x_2)$ if $x_1 \neq x_2$.
- We have established that the function $r : E \to \mathbb{Q}$ is injective, thus
$$\mathrm{card}(E) \leq \mathrm{card}(\mathbb{Q}). \qquad \square$$

Regular measures

If $F : \mathbb{R} \to \mathbb{R}$ is an increasing, right-continuous function on $\mathbb{R}$, then μ_F is the Borel measure given by the relation

$$\mu_F((a, b]) = F(b) - F(a).$$

The term almost everywhere will always refer to Lebesgue measure.

Definition (Regular measures)

A Borel measure ν on $\mathbb{R}^d$ is called **regular** if

1. $\nu(K) < \infty$ for every compact $K \subseteq \mathbb{R}^d$,
2. $\nu(E) = \inf\{\nu(U) : E \subseteq U, \text{ and } U \text{ is open}\}$ for every $E \in \mathrm{Bor}(\mathbb{R}^d)$.

Remark

In fact, (ii) is implied by (i), and this definition of regular measures coincides with the regularity discussed at the beginning of the course.

Differentiability of monotonic functions

Theorem (A)

Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and let

$$G(x) = F(x+) = \lim_{t \searrow x} F(t).$$

Then F and G are differentiable a.e. and $F' = G'$ a.e., and F and G are measurable on $\mathbb{R}$.

Proof. Observe that G is increasing and right-continuous, and $G = F$ except perhaps where F is discontinuous. Moreover,

$$G(x+h) - G(x) = \begin{cases} \mu_G((x, x+h]) & \text{if } h > 0, \\ -\mu_G((x+h, x]) & \text{if } h < 0. \end{cases}$$

The families $(x - r, x]$ and $(x, x + r]$ shrink nicely to x as $r = |h| \to 0$.

Proof

Last time we proved the following theorem:

Theorem (Lebesgue differentiation theorem for Borel measures)

Let ν be a regular signed or complex Borel measure on $\mathbb{R}^d$ and let $d\nu = d\mu + f\, d\lambda_d$ be the Lebesgue–Radon–Nikodym decomposition. Then for λ_d-a.e. $x \in \mathbb{R}^d$ we have

$$\lim_{r\to 0} \frac{\nu(E_r)}{\lambda_d(E_r)} = f(x), \quad \text{and} \quad \lim_{r\to 0} \frac{\mu(E_r)}{\lambda_d(E_r)} = 0,$$

for every family $(E_r)_{r>0} \subseteq \mathbb{R}^d$ that shrinks nicely to $x \in \mathbb{R}^d$.

- Applying this theorem to μ_G we see that $G'(x)$ exists for a.e. $x \in \mathbb{R}$.
- It remains to show that if $H = G - F$, then $H' = 0$ exists a.e. on $\mathbb{R}$.

Proof

- Let $E = \{x_j : j \in \mathbb{N}\}$ be an enumeration of points for which $H \neq 0$.
- Then $H(x_j) > 0$ and for $A \subseteq \mathbb{R}$ define a Borel measure

$$\mu(A) = \sum_{j\in\mathbb{N}} H(x_j)\delta_{x_j}(A), \quad \text{where} \quad \delta_{x_j}(x) = \begin{cases} 1 & \text{if } x = x_j, \\ 0 & \text{otherwise.} \end{cases}$$

- Observe that $\mu((-N, N)) < \infty$ for any $N \in \mathbb{N}$. In particular, μ is finite on compact subsets of $\mathbb{R}$, thus regular.
- Note also that $\mu \perp \lambda_1$ since $\lambda_1(E) = \lambda_1(E^c) = 0$. But then again by the Lebesgue differentiation theorem for Borel measures we may conclude

$$\left|\frac{H(x+h) - H(x)}{h}\right| \leq \frac{\mu((x - 2|h|, x + 2|h|))}{|h|} \xrightarrow[h\to 0]{} 0$$

for a.e. $x \in \mathbb{R}$. Thus $H'(x) = 0$ for a.e. $x \in \mathbb{R}$. $\square$

Lebesgue theorem

Theorem (Lebesgue theorem)

Suppose that $a \leq b$ in $\mathbb{R}$, and that $F : [a, b] \to \mathbb{R}$ is a non-decreasing function. Then $\int_a^b F'(x)dx$ exists and

$$\int_a^b F'(x)dx \leq F(b) - F(a). \qquad (*)$$

Remark

- In general, equality in (*) might not hold, even if we assume that F is a continuous function.
- In order to see this, let $F : [0, 1] \to [0, 1]$ be the Cantor–Lebesgue function. This is continuous increasing function such that $F(1) = 1$ and $F(0) = 0$. Moreover, F is constant on each interval of the complement of the Cantor set $\mathcal{C}$. Since $\lambda_1(\mathcal{C}) = 0$, we find that $F'(x) = 0$ a.e. on $[0, 1]$, yielding that equality in (*) cannot hold.

Proof

Proof. The result is trivial if $a = b$; let us suppose that $a < b$. Let us extend the function F from $[a, b]$ to $\mathbb{R}$ by setting

$$F(x) = \begin{cases} F(a) & \text{if } x \in (-\infty, a), \\ F(x) & \text{if } x \in [a, b], \\ F(b) & \text{if } x \in (b, \infty). \end{cases}$$

By Theorem (A) we see that F' is measurable and defined a.e. on $\mathbb{R}$. Let

$$F_n(x) = \frac{F(x + 1/n) - F(x)}{1/n} \quad \text{for} \quad x \in \mathbb{R},\ n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} F_n(x) = F'(x)$ for a.e. $x \in \mathbb{R}$. Since $F_n(x) \geq 0$ by Fatou's lemma we obtain that

$$\int_a^b F'(x)dx \leq \liminf_{n\to\infty} \int_a^b F_n(x)dx.$$

Proof

By translation invariance of Lebesgue measure we note that

$$\begin{aligned}
\int_a^b F_n(x)dx &= n\int_a^b F(x+1/n)dx - n\int_a^b F(x)dx \\
&= n\int_{a+1/n}^{b+1/n} F(x)dx - n\int_a^b F(x)dx \\
&= n\int_b^{b+1/n} F(x)dx - n\int_a^{a+1/n} F(x)dx \\
&\leq n\big(\lambda_1((b, b+1/2))F(b) - \lambda_1((a, a+1/2))F(a)\big) \\
&= F(b) - F(a).
\end{aligned}$$

To justify the inequality it suffices to note that $F(a) \leq F(x) \leq F(b)$ for all $x \in \mathbb{R}$. This completes the proof of the theorem. $\square$

Total variation

Definition (Total variation)

If $F : \mathbb{R} \to \mathbb{C}$ and $x \in \mathbb{R}$ we define the **total variation function** of F by

$$T_F(x) = \sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, \quad -\infty < x_0 < \ldots < x_n = x \right\}.$$

Remarks

- The sums in T_F are made bigger if we add more subdivision points.
- If $a < b$ the definition of $T_F(b)$ is unaffected if we assume that a is always one of the subdivision point. Then

$$\begin{aligned} & T_F(b) - T_F(a) \\ & \qquad = \sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, \quad a = x_0 < \ldots < x_n = b \right\}. \end{aligned}$$

- Thus T_F is an increasing function with values in $[0, \infty]$.

Bounded variation

Definition (Bounded variation)

If $T_F(\infty) = \lim_{x\to\infty} T_F(x) < \infty$ we say that F is of **bounded variation** on $\mathbb{R}$ and define

$$\mathrm{BV} = \{F : \mathbb{R} \to \mathbb{C} : T_F(\infty) < \infty\}.$$

The **total variation** of F on $[a, b]$ is defined by

$$T_F(b) - T_F(a) = \sup\Big\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, \quad a = x_0 < \ldots < x_n = b \Big\}.$$

It depends only on the values of F on $[a, b]$ and we may set

$$\mathrm{BV}([a, b]) = \{F : [a, b] \to \mathbb{C} : T_F(b) - T_F(a) < \infty\}.$$

If $F \in \mathrm{BV}$, then $F \in \mathrm{BV}([a, b])$ for all $a < b$. If $F \in \mathrm{BV}([a, b])$, then setting $F(x) = F(a)$ for $x < a$ and $F(x) = b$ for $x > b$ we have $F \in \mathrm{BV}$.

Examples

Examples

- If $F : \mathbb{R} \to \mathbb{R}$ is bounded and increasing then $f \in \mathrm{BV}$ and
$$T_F(x) = F(x) - F(-\infty).$$
- If $F, G \in \mathrm{BV}$, then $aF + bG \in \mathrm{BV}$ for all $a, b \in \mathbb{C}$.
- If F is differentiable and F' is bounded then $F \in \mathrm{BV}([a,b])$ for $-\infty < a < b < \infty$ by the mean value theorem.
- If $F(x) = \sin(x)$, then $F \in \mathrm{BV}([a,b])$ for $-\infty < a < b < \infty$, but $F \notin \mathrm{BV}$.
- Let
$$F(x) = \begin{cases} x \sin(x^{-1}) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$
Then $F \notin \mathrm{BV}([a,b])$ for $a \leq 0 < b$ or $a < 0 \leq b$.

Useful lemma

Lemma

If $F \in \mathrm{BV}$ and $F : \mathbb{R} \to \mathbb{R}$, then $F = \frac{1}{2}\left((T_F + F) - (T_F - F)\right)$, and $T_F + F$ and $T_F - F$ are increasing.

Proof. If $x < y$ and $\varepsilon > 0$ choose $x_0 < \ldots < x_n = x$ such that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq T_F(X) - \varepsilon.$$

Then $\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| + |F(y) - F(x)|$ is an approximation for $T_F(y)$ and $F(y) = (F(y) - F(x)) + F(x)$. Therefore,

$$\begin{aligned} T_F(y) \pm F(y) \geq & \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| + |F(y) - F(x)| \\ & \pm (F(y) - F(x)) \pm F(x) \geq T_F(x) - \varepsilon \pm F(x). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain $T_F(y) \pm F(y) \geq T_F(x) \pm F(x)$. $\square$

Properties of BV functions and Jordan decomposition

Theorem (B)

(a) *$F \in BV$ iff $\mathrm{Re}(F) \in \mathrm{BV}$ and $\mathrm{Im}(F) \in \mathrm{BV}$.*

(b) *If $F : \mathbb{R} \to \mathbb{R}$ then $F \in \mathrm{BV}$ iff F is the difference of two bounded increasing functions. We have the Jordan decomposition of $F \in \mathrm{BV}$*

$$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F),$$

into **positive** $\frac{1}{2}(T_F + F)$ *and* **negative** $\frac{1}{2}(T_F - F)$ *variations.*

(c) *If $F \in \mathrm{BV}$, then*

$$F(x+) = \lim_{y \searrow x} F(y) \quad \text{and} \quad F(x-) = \lim_{y \nearrow x} F(y)$$

exist for all $x \in \mathbb{R}$, as do $F(\pm\infty) = \lim_{y \to \pm\infty} F(y)$.

(d) *If $F \in \mathrm{BV}$ the set of points for which F is discontinuous if countable.*

(e) *If $F \in \mathrm{BV}$ and $G(x) = F(x+)$ then F' and G' exist and are equal a.e.*

Proof

Proof of (a). It is obvious since $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$. $\square$

Proof of (b). If $F = F_1 - F_2$ and F_1, F_2 are increasing and bounded, then $F_1, F_2 \in \mathrm{BV}$ and so $F = F_1 - F_2 \in \mathrm{BV}$. By the previous lemma we have $F = F_1 - F_2$, where $F_1 = \frac{1}{2}(T_F + F)$ and $F_2 = \frac{1}{2}(T_F - F)$, and F_1, F_2 are both increasing.

- Since $T_F(y) \pm F(y) \geq T_F(x) \pm F(x)$ for $y > x$, then

$$\mp F(y) \pm F(x) \leq T_F(y) - T_F(x) \leq T_F(\infty) - T_F(-\infty) < \infty,$$

  so that F, and hence $T_F \pm F$ is bounded. $\square$

Proof of (c). If $F \in \mathrm{BV}$, then F is a difference of two bounded increasing functions, thus the desired limits exist. $\square$

Proof of (d). If $F \in \mathrm{BV}$, then F is a difference of two bounded increasing functions, thus the set of discontinuities of F is countable. $\square$

Proof of (e). It follows from Theorem (A) and items (a) and (b). $\square$

Connection between BV and complex Borel measures

Since $x^+ = \max\{x, 0\} = \frac{1}{2}(|x| + x)$ and $x^- = \max\{-x, 0\} = \frac{1}{2}(|x| - x)$, for $x \in \mathbb{R}$ we have

$$\frac{1}{2}(T_F \pm F)(x) = \sup\left\{ \sum_{j=1}^{n} (F(x_j) - F(x_{j-1}))^{\pm} : x_0 < \ldots < x_N = x \right\} \\ \pm \frac{1}{2} F(-\infty).$$

Items (a) and (b) from Theorem (B) lead to the connection between BV and the space of complex Borel measures on $\mathbb{R}$. To make this precise we introduce the space

$$\mathrm{NBV} = \{F \in \mathrm{BV} : F \text{ is right-continuous and } F(-\infty) = 0\}$$

of normalize BV functions.

Observation

Observation

If $F \in \mathrm{BV}$ then

$$G(x) = F(x+) - F(-\infty) \in \mathrm{NBV}$$

and $G' = F'$ a.e. on $\mathbb{R}$. It only suffices to show that $G \in \mathrm{BV}$. Using the Jordan decomposition for F, we may write $F = F_1 - F_2$, where F_1, F_2 are increasing and bounded functions. Then

$$G(x) = F_1(x+) - (F_2(x+) - F(-\infty)),$$

which is the difference of two bounded increasing functions, thus $G \in \mathrm{BV}$.

Lemma (C)

If $F \in \mathrm{BV}$ then $T_F(-\infty) = 0$. If F is also right-continuous, then so is T_F.

Proof

Proof. If $\varepsilon > 0$ and $x \in \mathbb{R}$ choose $x_0 < \ldots < x_n = x$ so that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq T_F(x) - \varepsilon.$$

Hence $T_F(x) - T_F(x_0) \geq T_F(x) - \varepsilon$, which ensures that $T_F(-\infty) = 0$, since $T_F(y) \leq T_F(x_0) \leq \varepsilon$ for all $y \leq x_0$.

Suppose that F is right–continuous. Given $x \in \mathbb{R}$ and $\varepsilon > 0$ let us define $\alpha = T_F(x+) - T_F(x)$, and choose $\delta > 0$ so that $|F(x+h) - F(x)| < \varepsilon$ and $T_F(x+h) - T_F(x+) < \varepsilon$ when $0 < h < \delta$.

For any such h there exist $x = x_0 < \ldots < x_n = x + h$ such that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq \frac{3}{4}(T_F(x+h) - T_F(x)) \geq \frac{3}{4}\alpha,$$

since $\alpha = T_F(x+) - T_F(x)$.

Proof

Hence

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq \frac{3}{4}\alpha - |F(x_1) - F(x_0)| \geq \frac{3}{4}\alpha - \varepsilon.$$

Likewise, there exist $x = t_0 < \ldots < t_m = x_1$ such that

$$\sum_{j=1}^{m} |F(t_j) - F(t_{j-1})| \geq \frac{3}{4}\alpha$$

and hence

$$\begin{aligned}\alpha + \varepsilon > T_F(x+h) - T_F(x) \\ &\geq \sum_{j=1}^{m} |F(y) - F(t_{j-1})| + \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \\ &\geq \frac{3}{2}\alpha - \varepsilon,\end{aligned}$$

since $\alpha = T_F(x+) - T_F(x)$. Thus $\alpha < 4\varepsilon$ which gives $\alpha = 0$. $\square$

Correspondence between NBV and complex measures

Theorem (D)

If μ is a complex Borel measure on $\mathbb{R}$ and $F(x) = \mu((-\infty, x])$, then $F \in \mathrm{NBV}$. Conversely, if $F \in \mathrm{NBV}$, there exists a unique complex Borel measure μ_F such that $F(x) = \mu_F((-\infty, x])$.

Proof. If μ is a complex measure we have

$$\mu = \mu_1^+ - \mu_1^- + i(\mu_2^+ - \mu_2^-),$$

where $\mu_j^\pm$ for $j \in \{1, 2\}$ are finite measures.

- If $F_j^\pm(x) = \mu_j^\pm((-\infty, x])$, then $F_j^\pm$ is increasing and right continuous.
- $F_j^\pm(-\infty) = 0$, and $F_j^\pm(+\infty) = \mu_j^\pm(\mathbb{R}) < \infty$, thus $F_j^\pm$ are bounded.

By Theorem (B), see item (a) and (b), the function

$$F = F_1^+ - F_1^- + i(F_2^+ - F_2^-) \in \mathrm{NBV}.$$

Proof

Conversely, suppose that $F \in \mathrm{NBV}$.

- By item (a) of Theorem (B) we have $F = F_1 + iF_2$ for some $F_1, F_2 \in \mathrm{NBV}$.
- By item (b) of Theorem (B) we have

$$F = F_1^+ - F_1^- + i(F_2^+ - F_2^-), \quad \text{where} \quad F_j^\pm = F_j \pm T_{F_j}.$$

- By Lemma (C) we conclude that $F_j^\pm \in \mathrm{NBV}$.
- Now each $F_j^\pm$ gives rise to a measure $\mu_j^\pm$ so that

$$F(x) = \mu_F((-\infty, x]),$$

where

$$\mu_F = \mu_1^+ - \mu_1^- + i(\mu_2^+ - \mu_2^-). \qquad \square$$

Key Proposition

Proposition (E)

If $F \in \mathrm{NBV}$, then $F' \in L^1(\mathbb{R})$. Moreover, $\mu_F \perp \lambda$ iff $F' = 0$ a.e. on $\mathbb{R}$, and $\mu_F \ll \lambda$ iff $F(x) = \int_{\infty}^{x} F'(x)dx$.

Proof. Since $F \in \mathrm{NBV}$, then by Theorem (D) there exists a regular complex Borel measure μ_F such that $F(x) = \mu_F((-\infty, x])$.

- By the Lebesgue–Radon–Nikodym theorem: $\mu_F = (\mu_F)_s + (\mu_F)_a$.
- By the Lebesgue differentiation theorem for Borel measures we have

$$F'(x) = \lim_{r\to 0} \frac{\mu_F(E_r)}{\lambda(E_r)} = \lim_{r\to 0} \frac{(\mu_F)_a(E_r)}{\lambda(E_r)} = \frac{d\mu_F}{d\lambda}(x), \quad \text{and} \quad \lim_{r\to 0} \frac{(\mu_F)_s(E_r)}{\lambda(E_r)} = 0,$$

a.e. on $\mathbb{R}$, where $E_r = (x, x+r]$ or $(x-r, x]$. Hence

$$\mu_F \perp \lambda \ \text{ iff } \ F' = 0 \text{ a.e.}, \quad \text{and} \quad \mu_F \ll \lambda \ \text{ iff } \ F(x) = \int_{-\infty}^{x} F'(y)dy,$$

since μ_F is finite measure and $F' = \frac{d\mu_F}{d\lambda}$ is the Radon–Nikodym derivative we conclude that $F' \in L^1(\mathbb{R})$. □

Absolutely continuous functions

Definition (Absolutely continuous functions)

A function $F : \mathbb{R} \to \mathbb{C}$ is called **absolutely continuous (a.c.)** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite set of disjoint intervals $(a_1, b_1), \dots, (a_N, b_N)$, we have

$$\sum_{j=1}^{N}(b_j - a_j) < \delta \quad \Longrightarrow \quad \sum_{j=1}^{N} |F(b_j) - F(a_j)| < \varepsilon.$$

Remarks

- F is said to be **(a.c.)** on $[a, b]$ if this condition is satisfies when the intervals $(a_j, b_j) \subseteq [a, b]$.
- Clearly, if F is **(a.c.)** then F is uniformly continuous, (take $N = 1$).
- If F is differentiable and $|F'| \leq M$, then F is **(a.c.)**, and more generally any function satisfying Lipschitz condition is **(a.c.)**.

Absolute continuity for measures

The concepts of absolutely continuous measures and functions are very close to each other. Recall:

Theorem (The ε-δ criterion for absolute continuity)

Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$, then $\nu \ll \mu$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) < \delta \quad \Longrightarrow \quad |\nu|(E) < \varepsilon.$$

Corollary

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in L^1(X, \mu)$, then for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\mu(E) < \delta \quad \Longrightarrow \quad \Big| \int_E f d\mu \Big| < \varepsilon.$$

We will write $d\nu = f d\mu$ to express the relationship $\nu(E) = \int_E f d\mu$.

Absolute continuity for functions and measures

Proposition (F)

If $F \in \mathrm{NBV}$, then F is **(a.c.)** *iff $\mu_F \ll \lambda$.*

Proof ($\Longleftarrow$). Since $F \in \mathrm{NBV}$, then by Theorem (D) there exsist a regular complex Borel measure μ_F such that $F(x) = \mu_F((-\infty, x])$. If $\mu_F \ll \lambda$, by the ε-δ criterion for absolute continuity, for every $\varepsilon > 0$ there is $\delta > 0$ such that $|\mu_F(E)| < \varepsilon$ if $\lambda(E) < \delta$ where $E = \bigcup_{j=1}^{N}(a_j, b_j]$, thus F is **(a.c.)**. □

Proof ($\Longrightarrow$). To prove the converse let E be a Borel set with $\lambda(E) = 0$. If ε, δ are as in the definition of (a.c.) of F then we can find open sets $U_1 \supseteq U_2 \supseteq \ldots \supseteq E$ such that $\lambda(U_j) < \delta$ and $\mu_F(U_j) \xrightarrow[j\to\infty]{} \mu_F(E)$. Each U_j is a disjoint union of open intervals $(a_j^k, b_j^k) \subseteq U_j$ and

$$\sum_{k=1}^{N} |\mu_F((a_j^k, b_j^k))| \leq \sum_{k=1}^{N} |F(b_j^k) - F(a_j^k)| < \varepsilon \quad \text{for all} \quad N \in \mathbb{N}.$$

Letting $N \to \infty$ we obtain $|\mu_F(U_1)| < \varepsilon$ and hence $|\mu_F(E)| \leq \varepsilon$, thus $\mu_F(E) = 0$, since ε was arbitrary. This proves that $\mu_F \ll \lambda$ as desired. □

Important corollary

Corollary (G)

If $f \in L^1(\mathbb{R})$ then $F(x) = \int_{-\infty}^{x} f(t)dt \in \mathrm{NBV}$ and F is **(a.c.)**, *and $f = F'$ a.e. on $\mathbb{R}$. Conversely, if $F \in \mathrm{NBV}$ and F is* **(a.c.)**, *then $F' \in L^1(\mathbb{R})$ and $F(x) = \int_{-\infty}^{x} F'(t)dt$.*

Proof. If $f \in L^1(\mathbb{R})$ then $F(-\infty) = 0$ and $f = f_1^+ - f_1^- + i(f_2^+ - f_2^-)$, where $f_j^\pm \geq 0$. Then $F_j^\pm(x) = \int_{-\infty}^{x} f_j^\pm(t)dt$ is increasing, thus $F \in \mathrm{BV}$ by item (a) of Theorem (B). Moreover, F is right-continuous, so $F \in \mathrm{NBV}$.

- Since $F \in \mathrm{NBV}$, then by Theorem (D) there exsist a regular complex Borel measure μ_F such that $F(x) = \mu_F((-\infty, x])$.
- Since $F(x) = \int_{-\infty}^{x} f(t)dt$ then $\mu_F \ll \lambda$, which shows that F is **(a.c.)** by Proposition (F).
- By Proposition (E) we deduce $F(x) = \int_{-\infty}^{x} F'(t)dt$, since $\mu_F \ll \lambda$ and $F \in \mathrm{NBV}$, and consequently $F' = f$ a.e. on $\mathbb{R}$.

Proof

Conversely suppose that $F \in \mathrm{NBV}$ and F is **(a.c.)**, then we have $\mu_F \ll \lambda$ by Proposition (F). By Proposition (E) we readily see that $F' \in L^1(\mathbb{R})$ and $F(x) = \int_{-\infty}^{x} F'(t)dt$, and the proof is completed. □

Lemma (H)

If F is **(a.c.)** *on $[a, b]$ then $F \in \mathrm{BV}([a, b])$.*

Proof. Let δ be as in the definition of **(a.c.)** of the function F with $\varepsilon = 1$ and $N = \lfloor \delta^{-1}(b - a) + 1 \rfloor$. If $a = x_0 < \ldots < x_n = b$. By inserting more subdivision points if necessary we can collect the intervals (x_{j-1}, x_j) into at most N groups of consecutive intervals such that the sum of the lengths in each groups is less that δ. Then

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \leq N,$$

since the sum over each group is at most 1 and hence the total variation of F on $[a, b]$ is at most N. □

Fundamental theorem of calculus for Lebesgue integrals

Theorem (The fundamental theorem of calculus for Lebesgue integrals)

If $-\infty < a < b < \infty$ and $F : [a, b] \to \mathbb{C}$ the following are equivalent:

(a) *F is absolutely continuous on $[a, b]$.*

(b) *There is $f \in L^1([a, b])$ such that*

$$F(x) - F(a) = \int_a^x f(t)dt.$$

(c) *F is differentiable a.e. on $[a, b]$, and $F' \in L^1([a, b])$ and*

$$F(x) - F(a) = \int_a^x F'(t)dt.$$

Proof (a) $\Longrightarrow$ (c). Let $G(x) = F(x) - F(a)$ then $G(a) = 0$. We extend $G(x)$ to the entire $\mathbb{R}$ by setting $G(x) = 0$ for all $x < a$ and $G(x) = G(b)$ for all $x > b$. Hence G is absolutely continuous on $\mathbb{R}$.

Proof

- Since F is absolutely continuous on $[a,b]$, then $f \in \mathrm{BV}([\mathrm{a},\mathrm{b}])$ by Lemma (H) and consequently $G \in \mathrm{NBV}$ and G is **(a.c.)**.
- By the previous corollary we deduce that $G' = F' \in L^1(\mathbb{R})$ and

$$F(x) - F(a) = G(x) = \int_{-\infty}^{x} G'(t)dt = \int_{a}^{x} F'(t)dt. \qquad \square$$

Proof (c) $\Longrightarrow$ (b). We now see that this implication is obvious. $\square$

Proof (b) $\Longrightarrow$ (a). We set $f(x) = 0$ for $t \in [a,b]^c$ and apply the previous corollary. Then

$$F(x) - F(a) = G(x) = \int_{-\infty}^{x} f(t)dt \in \mathrm{NBV},$$

and G is **(a.c.)** on $\mathbb{R}$, thus F must be **(a.c.)** on $[a,b]$. $\square$

Integration by parts

Theorem (Integration by parts)

Let f and g be two real-valued absolutely continuous functions on $[a,b]$. Then fg' and $f'g$ are in $L^1([a,b])$ and for every $x \in [a,b]$ we have

$$\int_a^x f'(t)g(t)dt = f(x)g(x) - f(a)g(a) - \int_a^x f(t)g'(t)dt \quad \text{for} \quad x \in [a,b]$$

Proof. The absolute continuity of f and g implies that of fg. By the previous theorem f', g' and $(fg)'$ exist a.e. on $[a,b]$ and all belong to $L^1([a,b])$. Moreover, $fg', f'g \in L^1([a,b])$. Thus for $x \in [a,b]$ we have

$$\begin{aligned} f(x)g(x) - f(a)g(a) &= \int_a^x (f(t)g(t))' dt \\ &= \int_a^x f(t)'g(t)dt + \int_a^x f(t)g(t)'dt. \end{aligned}$$

This gives the desired result. □

Change of variables formula for **(a.c.)** functions

Theorem (Change of variables formula for **(a.c.)** functions)

Suppose that $\varphi : [a, b] \to [\alpha, \beta]$ *is an increasing* **(a.c.)** *function such that* $\varphi(a) = \alpha$ *and* $\varphi(b) = \beta$. *Then for every* $f \in L^1([\alpha, \beta])$ *one has*

$$\int_\alpha^\beta f(t)dt = \int_a^b f(\varphi(t))\varphi'(t)dt. \qquad (*)$$

Proof. We first verify (*) for $f = \mathbb{1}_{(\alpha_0,\beta_0)}$, where $(\alpha_0, \beta_0) \subset [\alpha, \beta]$. Since φ is continuous and increasing function thus $\varphi^{-1}[(\alpha_0, \beta_0)] = (a_0, b_0)$ for some $(a_0, b_0) \subset [a, b]$ and $\varphi(a_0) = \alpha_0$ and $\varphi(b_0) = \beta_0$. Then we see that

$$\int_\alpha^\beta f(t)dt = \beta_0 - \alpha_0 = \varphi(b_0) - \varphi(a_0) = \int_{a_0}^{b_0} \varphi'(t)dt = \int_a^b f(\varphi(t))\varphi'(t)dt,$$

which is precisely (*) with $f = \mathbb{1}_{(\alpha_0,\beta_0)}$. Now we can quite easily pass from characteristic functions of open intervals to characteristic functions of Borel and then Lebesgue measurable sets, and then by standard approximation arguments to all $f \in L^1([\alpha, \beta])$. Justify this! $\square$