Lecture 21

Borel and Radon measures
Riesz representation theorem

Graduate Real Analysis

Fall Semester

Borel measures

Notations

- Unless otherwise stated, X is a **topological space**.
- $\mathcal{U}_X = \mathrm{Open}(X)$ is the is the collection of all **open sets** in X.
- $\mathcal{F}_X = \mathrm{Closed}(X)$ is the is the collection of all **closed sets** in X.
- $\mathcal{K}_X = \mathrm{Compact}(X)$ is the collection of all **compact sets** in X,
- $\mathcal{B}_X = \mathrm{Bor}(X) = \sigma(\mathcal{U}_X) = \sigma(\mathcal{F}_X)$ is the **Borel σ-algebra** on X.

Borel measures

A measure μ defined on a Borel σ-algebra $\mathcal{B}_X$ is called a **Borel measure**. A pair $(X, \mathcal{B}_X)$ is called a **Borel space**, whereas a triple $(X, \mathcal{B}_X, \mu)$ is called a **Borel measure space**.

Regularity of Borel measures: finite case

Theorem

Let $(X, \mathcal{B}_X, \mu)$ be a finite Borel measure space such that every closed set in X is a G_δ set. Then for every $E \in \mathcal{B}_X$ and every $\varepsilon > 0$ there are two sets: an open set $G \supseteq E$ and a closed set $F \subseteq E$ such that

$$\mu(E \setminus F) \leq \varepsilon \qquad \textit{and} \qquad \mu(G \setminus E) \leq \varepsilon.$$

Remark

If (X, d) is a metric set then every closed set in X is a G_δ set in X.

Proof of the Remark.

Let F be a closed set in X and observe that

$$F = \bigcap_{n \in \mathbb{N}} \left\{ x \in X : d(x, F) < \frac{1}{n} \right\},$$

as desired. □

Proof of the theorem

Proof. Let $\mathcal{M}$ be the set of all Borel sets $E \in \mathcal{B}_X$ with the property that for every $\varepsilon > 0$ there are two sets: an open set $G \supseteq E$ and a closed set $F \subseteq E$ such that

$$\mu(E \setminus F) \leq \varepsilon \qquad \text{and} \qquad \mu(G \setminus E) \leq \varepsilon.$$

We show that $\mathcal{M}$ is a σ-algebra containing all open subsets of X. This in turn will imply that $\mathcal{M} = \mathcal{B}_X$, and the proof will be finished.

- $\mathcal{M}$ is closed under complements. Indeed, if $E \in \mathcal{M}$ then for every $\varepsilon > 0$ there are two sets: closed F and open G such that

$$F \subseteq E \subseteq G \quad \Longleftrightarrow \quad G^c \subseteq E^c \subseteq F^c \qquad \text{and}$$

$$\mu(E^c \setminus G^c) = \mu(G \setminus E) \leq \varepsilon \qquad \text{and} \qquad \mu(F^c \setminus E^c) = \mu(E \setminus F) \leq \varepsilon,$$

and we are done since G^c is closed and F^c is open.

Proof of the theorem

- $\mathcal{M}$ is closed under countable unions. Let $E_1, E_2, \ldots \in \mathcal{M}$ and let $E = \bigcup_{n\in\mathbb{N}} E_n$. For every $\varepsilon > 0$ and every $n \in \mathbb{N}$ there are closed F_n and open G_n sets such that $F_n \subseteq E_n \subseteq G_n$ and

$$\mu(G_n \setminus E_n) \le \varepsilon 2^{-n} \qquad \text{and} \qquad \mu(E_n \setminus F_n) \le \varepsilon 2^{-n-1}.$$

Define an open set $G = \bigcup_{n\in\mathbb{N}} G_n$ and note that $E \subseteq G$ and

$$\mu(G \setminus E) \le \sum_{n\in\mathbb{N}} \mu(G_n \setminus E) \le \sum_{n\in\mathbb{N}} \mu(G_n \setminus E_n) \le \sum_{n\in\mathbb{N}} \varepsilon 2^{-n} \le \varepsilon.$$

Consider $F_0 = \bigcup_{n\in\mathbb{N}} F_n$ and note that $F_0 \subseteq E$. By the continuity and finiteness of μ there is $m \in \mathbb{N}$ such that

$$\mu\left(F_0 \setminus \bigcup_{n=1}^{m} F_n\right) \le \varepsilon/2.$$

Proof of the theorem

Define a closed set $F = \bigcup_{n=1}^{m} F_n$ and note that $F \subseteq F_0 \subseteq E$ and $E \setminus F = E \cap F^c \cap F_0^c \cup E \cap F^c \cap F_0 \subseteq E \setminus F_0 \cup F_0 \setminus F$.

$$\mu(E \setminus F) \leq \mu(E \setminus F_0) + \mu(F_0 \setminus F) \leq \varepsilon,$$

since $\mu(F_0 \setminus F) = \mu\big(F_0 \setminus \bigcup_{n=1}^{m} F_n\big) \leq \varepsilon/2$, and

$$\mu(E \setminus F_0) \leq \sum_{n\in\mathbb{N}} \mu(E_n \setminus F_0) \leq \sum_{n\in\mathbb{N}} \mu(E_n \setminus F_n) \leq \sum_{n\in\mathbb{N}} \varepsilon 2^{-n-1} \leq \varepsilon/2.$$

- $\mathcal{M}$ contains open sets. Let $O \in \mathcal{U}_X$ be an open set. Fix $\varepsilon > 0$. Taking $G = O$ we see that $\mu(G \setminus O) = 0 < \varepsilon$. Now since every closed set in X is a G_δ set then $O = \bigcup_{n\in\mathbb{N}} F_n$ for some sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets. By the continuity and finiteness of μ there is $m \in \mathbb{N}$ such that $\mu\big(O \setminus \bigcup_{n=1}^{m} F_n\big) \leq \varepsilon$. Taking a closed set $F = \bigcup_{n=1}^{m} F_n$ we obtain the desired claim. $\square$

Regularity of Borel measures: σ-finite case

Theorem

Let $(X, \mathcal{B}_X, \mu)$ be a Borel measure space such that every closed set in X is a G_δ set. Assume that $X = \bigcup_{n\in\mathbb{N}} V_n$ for some open sets V_n such that $\mu(V_n) < \infty$ for every $n \in \mathbb{N}$. Then for every $E \in \mathcal{B}_X$ and every $\varepsilon > 0$ there are two sets: an open set $G \supseteq E$ and a closed set $F \subseteq E$ such that

$$\mu(E \setminus F) \leq \varepsilon \qquad \text{and} \qquad \mu(G \setminus E) \leq \varepsilon.$$

Remark

Suppose that $(X, \mathcal{B}_X, \mu)$ is a Borel measure space that is finite on every compact set. If additionally X is a locally compact Hausdorff space which is σ-compact (i.e. X is a countable union of compact sets) then there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of open sets with compact closures such that $\mathrm{cl}(U_n) \subseteq U_{n+1}$ for all $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} U_n$. Then we immediately obtain that

$$\mu(U_n) < \infty.$$

Proof of the theorem

Proof. We shall reduce the matters to the previous theorem. Decompose $X = \bigcup_{n\in\mathbb{N}} V_n$ for some open sets V_n such that $\mu(V_n) < \infty$ for all $n \in \mathbb{N}$.

- Let $E \in \mathcal{B}_X$, fix $\varepsilon > 0$ and define $E_n = E \cap V_n$ for every $n \in \mathbb{N}$.
- For each $n \in \mathbb{N}$ consider a σ-algebra $\mathcal{B}_X \restriction V_n = \{E \cap V_n : E \in \mathcal{B}_X\}$ with a measure $\mu_{\restriction V_n}$, which is a restriction of μ to $\mathcal{B}_X \restriction V_n$. Then $(V_n, \mathcal{B}_X \restriction V_n, \mu_{\restriction V_n})$ is a finite Borel measure space with the property that every closed set in V_n is a G_δ set. Therefore the conclusion of the previous theorem is true for this Borel measure space.
- For each $n \in \mathbb{N}$ we can find open sets $G_n \in \mathcal{B}_X \restriction V_n$, which are also open in X such that $G_n \supseteq E_n$ and $\mu(G_n \setminus E_n) \leq \varepsilon 2^{-n}$. Taking $G = \bigcup_{n\in\mathbb{N}} G_n$ we see that

$$\mu(G \setminus E) \leq \sum_{n\in\mathbb{N}} \mu(G_n \setminus E) \leq \sum_{n\in\mathbb{N}} \mu(G_n \setminus E_n) \leq \sum_{n\in\mathbb{N}} \varepsilon 2^{-n} \leq \varepsilon.$$

- Applying the same reasoning with E^c in place of E we obtain $\mu(E \setminus F) = \mu(E \setminus G^c) = \mu(G \setminus E^c) \leq \varepsilon$ with a closed $F = G^c$. $\square$

Outer regular, inner regular and regular measures

Definition

Let $(X, \mathcal{B}_X, \mu)$ be a Borel measure space.

- We say that μ is **outer regular** for $E \in \mathcal{B}_X$ if

$$\mu(E) = \inf\{\mu(U) : U \supseteq E \text{ and } U \in \mathcal{U}_X\}.$$

- We say that μ is **inner regular** for $E \in \mathcal{B}_X$ if

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \in \mathcal{K}_X\}.$$

- We say that μ is **regular** for $E \in \mathcal{B}_X$ if μ is both outer and inner regular for $E \in \mathcal{B}_X$.
- Let $\mathcal{V} \subseteq \mathcal{B}_X$ be an arbitrary collection. We say that μ is outer regular for $\mathcal{V}$, inner regular for $\mathcal{V}$, or regular for $\mathcal{V}$ according as μ is outer regular, inner regular, or regular for every $E \in \mathcal{V}$.

Regularity of Borel measures

Corollary

Let $(X, \mathcal{B}_X, \mu)$ be a σ-finite Borel measure space. Suppose that

- μ *is* **outer regular** *for $E \in \mathcal{B}_X$, i.e.*

$$\mu(E) = \inf\{\mu(U) : U \supseteq E \text{ and } U \in \mathcal{U}_X\}.$$

- μ *is μ is* **inner regular** *for $E \in \mathcal{B}_X$, i.e.*

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \in \mathcal{K}_X\}.$$

Then

(a) *For every $E \in \mathcal{B}_X$ and every $\varepsilon > 0$ there exists a closed set C and an open set U such that $C \subseteq E \subseteq U$ and $\mu(U \setminus C) \leq \varepsilon$.*

(b) *For every $E \in \mathcal{B}_X$ there exists an F_σ set F and a G_δ set G such that $F \subseteq E \subseteq G$ and $\mu(G \setminus F) = 0$.*

Radon measures

Definition

A measure μ on a Borel space $(X, \mathcal{B}_X)$ is called a Radon measure if μ is:

- **Outer regular** for $\mathcal{B}_X$:

$$\mu(E) = \inf\{\mu(U) : U \supseteq E \text{ and } U \in \mathcal{U}_X\} \quad \text{for every} \quad E \in \mathcal{B}_X.$$

- **Inner regular** for $\mathcal{U}_X$:

$$\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ and } K \in \mathcal{K}_X\} \quad \text{for every} \quad U \in \mathcal{U}_X.$$

- **Finite** on $\mathcal{K}_X$:

$$\mu(K) < \infty \quad \text{for every} \quad K \in \mathcal{K}_X.$$

σ-finite Radon measures are regular

Theorem

Let X be a Hausdorff space. Any σ-finite Radon measure μ on a Borel space $(X, \mathcal{B}_X)$ is regular for $\mathcal{B}_X$. In particular, every finite Radon measure is regular for $\mathcal{B}_X$.

Proof. We have to show that $\mu(E) = \alpha_E$, where

$$\alpha_E = \sup\{\mu(K) : K \subseteq E \text{ and } K \in \mathcal{K}_X\}.$$

for every $E \in \mathcal{B}_X$. Suppose that $\mu(E) < \infty$.

- Note that $\alpha_E \leq \mu(E)$ since $\mu(K) \leq \mu(E)$ for every compact $K \subseteq E$.
- It suffices to prove that $\alpha_E \geq \mu(E)$. Fix $\varepsilon > 0$.
- By the outer regularity for $\mathcal{B}_X$ we find an open set $U \supseteq E$ such that

$$\mu(U) \leq \mu(E) + \varepsilon/2.$$

- By the inner regularity for $\mathcal{U}_X$ we find a compact set $F \subseteq U$ such that

$$\mu(F) \geq \mu(U) - \varepsilon/2.$$

Proof

- By the outer regularity for $\mathcal{B}_X$ again, since $\mu(U \setminus E) \leq \varepsilon/2$ we can also choose and open $V \supseteq U \setminus E$ such that $\mu(V) \leq \varepsilon/2$.
- Consider a compact set $K = F \setminus V$ and note that

$$K = F \cap V^c \subseteq F \cap (U^c \cup E) \subseteq E,$$

since $V^c \subseteq (U \cap E^c)^c = U^c \cup E$ and $F \cap U^c = \emptyset$.
- Furthermore, since $\mu(F) \geq \mu(U) - \varepsilon/2$ and $E \subseteq U$ we obtain

$$\alpha_E \geq \mu(K) = \mu(F) - \mu(F \cap V) \geq \mu(U) - \varepsilon/2 - \mu(V) \geq \mu(E) - \varepsilon,$$

yielding $\alpha_E \geq \mu(E)$ as desired.
- Suppose that $\mu(E) = \infty$ and $E = \bigcup_{j\in\mathbb{N}} E_j$, where $E_j \in \mathcal{B}_X$ and $E_j \subseteq E_{j+1}$ and $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$. Thus for any $N \in \mathbb{N}$ there is $j \in \mathbb{N}$ such that $\mu(E_j) > N$ and by the preceding argument we find a compact $K \subseteq E_j$ with $\mu(K) > N$, which ensures that $\alpha_E = \infty$. $\square$

Borel measures which are Radon

Theorem

Suppose that X is a locally compact Hausdorff space in which every open set is σ-compact (i.e. is a countable union of compact sets). Then every Borel measure on $(X, \mathcal{B}_X)$ that is finite on compact sets is Radon.

Proof. A few comments are in order.

- Note that every closed set is a G_δ set, since every open set is σ-compact in X.
- Since X is a locally compact Hausdorff space which is σ-compact, then there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of open sets with compact closures such that $\mathrm{cl}(U_n) \subseteq U_{n+1}$ for all $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} U_n$. Then we immediately obtain that

$$\mu(U_n) < \infty.$$

Proof

- Therefore μ is outer regular for $\mathcal{B}_X$:

$$\mu(E) = \inf\{\mu(U) : U \supseteq E \text{ and } U \in \mathcal{U}_X\} \quad \text{for every} \quad E \in \mathcal{B}_X.$$

- We prove that μ is inner regular for $\mathcal{U}_X$:

$$\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ and } K \in \mathcal{K}_X\} \quad \text{for every} \quad U \in \mathcal{U}_X.$$

  Indeed, every open set U is σ-compact. Let $U = \bigcup_{n\in\mathbb{N}} K_n$ for some increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets. If $\mu(U) < \infty$ then by the continuity of the measure μ for any $\varepsilon > 0$ we find $n \in \mathbb{N}$ such that $\mu(K_n) \geq \mu(U) - \varepsilon$, which impies $\mu(U \setminus K_n) \leq \varepsilon$ as desired. If $\mu(U) = \infty$ then for any $M > 0$ there exists $n \in \mathbb{N}$ such that $\mu(K_n) \geq M$, which also gives the desired claim.

- Invoking the last theorem we know that in σ-finite measure spaces inner regularity for $\mathcal{U}_X$ becomes regularity for $\mathcal{B}_X$. □