Lecture 22

Infinite product measures
Kolmogorov extension theorem

Graduate Real Analysis

Fall Semester

Basic fact on countable additivity

Proposition

Let μ be an additive finite set function on an algebra of sets $\mathcal{A}$. Then the following conditions are equivalent:

(a) *the function μ is countably additive,*

(b) *the function μ is continuous at zero in the following sense: if $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ is a decreasing sequence of sets ($A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$) with $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, then*

$$\lim_{n\to\infty} \mu(A_n) = 0.$$

Proof of (a) $\Longrightarrow$ (b). Let μ be countably additive and let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ decrease monotonically to the empty set. Set $B_n = A_n \setminus A_{n+1} \in \mathcal{A}$. The sets B_n are disjoint and their union is A_1. Note that

$$\mu(A_1) = \sum_{k\in\mathbb{N}} \mu(B_k) < \infty.$$

Proof

Hence
$$\lim_{n\to\infty}\mu(A_n)=\lim_{n\to\infty}\sum_{k=n}^{\infty}\mu(B_k)=0$$
and (b) follows.

Proof of (b) $\Longrightarrow$ (a). Let $(B_n)_{n\in\mathbb{N}}\subseteq\mathcal{A}$ be a sequence of disjoint sets such that $\bigcup_{n\in\mathbb{N}}B_n=B\in\mathcal{A}$. Define
$$A_n=B\setminus\bigcup_{k=1}^{n}B_k\in\mathcal{A}.$$
It is clear that $(A_n)_{n\in\mathbb{N}}\subseteq\mathcal{A}$ forms a decreasing sequence of sets such that $\bigcap_{n\in\mathbb{N}}A_n=\emptyset$. By (b) we have $\lim_{n\to\infty}\mu(A_n)=0$. By finite additivity this in turn means that
$$\lim_{n\to\infty}\sum_{k=1}^{n}\mu(B_k)=\mu(B).$$
Here we also used finiteness of μ. $\square$

Basic fact on product σ-algebras

Product σ-algebra

Let $(X_\alpha)_{\alpha\in A}$ be an indexed collection of nonempty sets, $X = \prod_{\alpha\in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ the coordinate maps. If $\mathcal{A}_\alpha$ is a σ-algebra on X_α for each $\alpha \in A$, **the product σ-algebra on X** is the σ-algebra generated by

$$\{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{A}_\alpha,\ \alpha \in A\}.$$

We denote this σ-algebra by $\bigotimes_{\alpha\in A} \mathcal{A}_\alpha$. In other words,

$$\bigotimes_{\alpha\in A} \mathcal{A}_\alpha = \sigma\left(\{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{A}_\alpha,\ \alpha \in A\}\right).$$

If $A = \{1, \ldots, n\}$, we write $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \ldots \otimes \mathcal{A}_n$.

Proposition

Proposition

One has

$$\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\mathcal{F}),$$

where

$$\mathcal{F} = \left\{ \bigcap_{j=1}^{k} \pi_{\alpha_j}^{-1}[E_{\alpha_j}] : E_{\alpha_1} \in \mathcal{A}_{\alpha_1}, \ldots, E_{\alpha_k} \in \mathcal{A}_{\alpha_k},\ \alpha_1, \ldots, \alpha_k \in A,\ k \in \mathbb{N} \right\}.$$

Proof. Let

$$\mathcal{E} = \{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{A}_\alpha,\ \alpha \in A\}.$$

- By the definition $\sigma(\mathcal{E}) = \bigotimes_{\alpha \in A} \mathcal{A}_\alpha$.
- Obviously $\mathcal{E} \subseteq \mathcal{F} \subseteq \sigma(\mathcal{F})$, which implies $\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{F})$.
- It is easy to see that $\mathcal{F} \subseteq \sigma(\mathcal{E})$, hence $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{E})$, which gives $\sigma(\mathcal{F}) = \sigma(\mathcal{E})$ as desired. □

Infinite products of measures

Let $\{(X_\alpha, \mathcal{A}_\alpha, \mu_\alpha) : \alpha \in A\}$ be a family of probability spaces.

Remark

- By the previous proposition we know that $\bigotimes_{\alpha\in A} \mathcal{A}_\alpha = \sigma(\mathcal{F})$, where

$$\mathcal{F} = \Big\{ \bigcap_{j=1}^{k} \pi_{\alpha_j}^{-1}[E_{\alpha_j}] : E_{\alpha_1} \in \mathcal{A}_{\alpha_1}, \dots, E_{\alpha_k} \in \mathcal{A}_{\alpha_k},\ \alpha_1, \dots, \alpha_k \in A,\ k \in \mathbb{N} \Big\}.$$

- In other words, $\bigotimes_{\alpha\in A} \mathcal{A}_\alpha$ is generated by sets of the form $\prod_{\alpha\in A} A_\alpha$, where only finitely many A_α may differ from X_α. Sets of such a form are called **cylidrical** or **cylinders**.
- Our goal is to define a **product measure** μ on $X = \prod_{\alpha\in A} X_\alpha$ paired with the σ-algebra $\bigotimes_{\alpha\in A} \mathcal{A}_\alpha$ such that

$$\mu(E) = \bigotimes_{j=1}^{k} \mu_{\alpha_j}(E_{\alpha_j}) \quad \text{if} \quad E = \bigcap_{j=1}^{k} \pi_{\alpha_j}^{-1}[E_{\alpha_j}] \in \mathcal{F}$$

Countable products

We start with a countable set A. Let

$$\mathcal{E}_n = \{C \times \prod_{j=1}^{\infty} X_{n+j} : C \in \bigotimes_{i=1}^{n} \mathcal{A}_i\}$$

- Then
$$\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{E}_n$$
is an algebra, and it is not difficult to show that
$$\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\mathcal{A}).$$
- We define a set function $\mu : \mathcal{A} \to [0, 1]$ by setting
$$\mu(A) = \mu_1 \otimes \ldots \otimes \mu_n(C) \quad \text{for} \quad A = C \times \prod_{j=1}^{\infty} X_{n+j} \in \mathcal{E}_n.$$
- We show that μ is well-defined on $\mathcal{A}$.

μ is well-defined on $\mathcal{A}$

- If $A = C \times \prod_{j=1}^{\infty} X_{n+j} \in \mathcal{E}_n$ is regarded as an element of $\mathcal{E}_k$ with $k > n$, then it can be written as a set

$$E = (C \times X_{n+1} \times \ldots \times X_k) \times \prod_{j=1}^{\infty} X_{k+j} \in \mathcal{E}_k.$$

Then we have

$$\mu(E) = \mu_1 \otimes \ldots \otimes \mu_k(C \times \prod_{j=n+1}^{k} X_j) = \mu_1 \otimes \ldots \otimes \mu_n(C) = \mu(A),$$

since

$$\mu_{n+1}(X_{n+1}) = \ldots = \mu_k(X_k) = 1.$$

- μ is countably additive on $\bigotimes_{i=1}^{n} \mathcal{A}_i$ for each $n \in \mathbb{N}$, thus it finitely additivity on $\mathcal{A}$.
- We now show that μ is a premeasure on $\mathcal{A}$.

Countable additivity of μ

Theorem

Let $\{(X_\alpha, \mathcal{A}_\alpha, \mu_\alpha) : \alpha \in A\}$ be a countable family of probability spaces. The set function μ on the algebra $\mathcal{A}$ is a premeasure and hence it can be uniquely extended to a measure on $\sigma(\mathcal{A}) = \bigotimes_{\alpha \in A} \mathcal{A}_\alpha$.

Proof. It suffices to show that μ is countably additive on the algebra $\mathcal{A}$. Then by the Carathéodory extension theorem it can be uniquely extended to a countably additive measure on $\sigma(\mathcal{A}) = \bigotimes_{\alpha \in A} \mathcal{A}_\alpha$.

- To prove that μ is countably additive on the algebra $\mathcal{A}$ we take a decreasing sequence $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ of sets such that $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$, we have to show that
$$\lim_{n \to \infty} \mu(A_n) = 0.$$
- Suppose for a contradiction that $\mu(A_n) > \epsilon > 0$ for all $n \in \mathbb{N}$. We will show that $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$, which is impossible!

Proof

- Let $\mathcal{A}^{(n)}$ denote the algebra of sets in $\prod_{i=n+1}^{\infty} X_i$ defined by analogy with $\mathcal{A}$ and let $\mu^{(n)}$ be the set function on $\mathcal{A}^{(n)}$ corresponding to the product of the measures $\mu_{n+1}, \mu_{n+2}, \ldots$ by analogy with μ.
- By the properties of finite products it follows that, for every set $A \in \mathcal{A}$ and every fixed $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} X_i$, the section

$$A^{x_1,\ldots,x_n} = \Big\{(z_{n+1}, z_{n+2}, \ldots) \in \prod_{i=n+1}^{\infty} X_i : (x_1, \ldots, x_n, z_{n+1}, \ldots) \in A\Big\}$$

belongs to $\mathcal{A}^{(n)}$ and the function

$$\prod_{i=1}^{n} X_i \ni (x_1, \ldots, x_n) \mapsto \mu^{(n)}(A^{x_1,\ldots,x_n})$$

is measurable with respect to $\bigotimes_{i=1}^{n} \mathcal{A}_i$.

Proof

- Let
$$B_1^k = \{x_1 \in X_1 : \mu^{(1)}(A_k^{x_1}) > \epsilon/2\}.$$
- Then $B_1^k \in \mathcal{A}_1$ and $\mu_1(B_1^k) > \epsilon/2$, which follows by Fubini's theorem for finite products and the inequality $\mu(A_k) > \epsilon$ for all $k \in \mathbb{N}$.
- Indeed, $A_k = C_m \times \prod_{j=m+1}^{\infty} X_j$ for some $m \in \mathbb{N}$, whence one has $\mu(A_k) = \bigotimes_{i=1}^{m} \mu_i(C_m)$. By Fubini's theorem we obtain
$$\epsilon < \mu(A_k) \leq \mu_1(B_1^k) + \frac{\epsilon}{2}\mu_1(X_1 \backslash B_1^k) \leq \mu_1(B_1^k) + \epsilon/2$$
which yields the necessary estimate.
- The sequence of sets $(B_1^k)_{k\in\mathbb{N}}$ is decreasing and
$$B_1 = \bigcap_{k\in\mathbb{N}} B_1^k \neq \emptyset,$$
since μ_1 is a measure on X_1 and $\mu_1(B_1^k) > \epsilon/2$ for all $k \in \mathbb{N}$.

Proof

- We fix an arbitrary point $x_1 \in B_1$ and repeat the described procedure for the decreasing sets $A_k^{x_1}$ in place of A_k. This is possible, since $\mu^{(1)}(A_k^{x_1}) > \epsilon/2$. We obtain a point $x_2 \in X_2$ such that $\mu^{(2)}(A_k^{x_1,x_2}) > \epsilon/4$ for all $k \in \mathbb{N}$. We continue this process inductively.
- After the n-th step we obtain $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} X_i$ such that $\mu^{(n)}(A_k^{x_1,\ldots x_n}) > \epsilon 2^{-n}$ for all $k \in \mathbb{N}$. Our construction can be continued, yielding a point $x = (x_1, \ldots, x_n, \ldots)$ belonging to all A_k.
- Indeed, fix $k \in \mathbb{N}$ and write $A_k = C_m \times \prod_{j=m+1}^{\infty} X_j$ for some $m \in \mathbb{N}$. The set $A_k^{x_1,\ldots,x_m} \neq \emptyset$, so there exists $(z_{m+1}, z_{m+2}, \ldots) \in \prod_{i=m+1}^{\infty} X_i$ such that

$$(x_1, \ldots, x_m, z_{m+1}, z_{m+2}, \ldots) \in A_k,$$

implying $(x_1, \ldots, x_m) \in C_m$ and since $(x_{m+1}, x_{m+2}, \ldots) \in \prod_{j=m+1}^{\infty} X_j$, thus $(x_1, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots) \in A_k$ and we are done. $\square$

Arbitrary products

Proposition

One has $\bigotimes_{\alpha\in A}\mathcal{A}_\alpha = \sigma(\mathcal{F})$, where

$$\mathcal{F} = \left\{ \bigcap_{j=1}^{\infty} \pi_{\alpha_j}^{-1}[E_{\alpha_j}] : E_{\alpha_1} \in \mathcal{A}_{\alpha_1}, E_{\alpha_2} \in \mathcal{A}_{\alpha_2}, \dots, \text{ and } (\alpha_j)_{j\in\mathbb{N}} \subseteq A \right\}.$$

In other words, $\bigotimes_{\alpha\in A}\mathcal{A}_\alpha$ is generated by sets of the form $\prod_{\alpha\in A} A_\alpha$, where only countably many A_α may differ from X_α.

Proof. By the definition $\sigma(\mathcal{E}) = \bigotimes_{\alpha\in A}\mathcal{A}_\alpha$, where

$$\mathcal{E} = \{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{A}_\alpha,\ \alpha \in A\}.$$

- Obviously $\mathcal{E} \subseteq \mathcal{F} \subseteq \sigma(\mathcal{F})$, which implies $\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{F})$.
- It is easy to see that $\mathcal{F} \subseteq \sigma(\mathcal{E})$, hence $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{E})$, which gives $\sigma(\mathcal{F}) = \sigma(\mathcal{E})$ as desired. $\square$

Extension to uncountable A

We fix an arbitrary uncountable index set A.

- By the previous proposition we know that $\bigotimes_{\alpha\in A}\mathcal{A}_\alpha = \sigma(\mathcal{F})$, where

$$\mathcal{F} = \Big\{ \bigcap_{j=1}^{\infty} \pi_{\alpha_j}^{-1}[E_{\alpha_j}] : E_{\alpha_1} \in \mathcal{A}_{\alpha_1}, E_{\alpha_2} \in \mathcal{A}_{\alpha_2}, \dots, \text{ and } (\alpha_j)_{j\in\mathbb{N}} \subseteq A \Big\}.$$

- Thus

$$\bigotimes_{\alpha\in A}\mathcal{A}_\alpha = \bigcup_{(\alpha_j)_{j\in\mathbb{N}}\subseteq A} \sigma(\mathcal{F}_{(\alpha_j)_{j\in\mathbb{N}}}),$$

  where

$$\mathcal{F}_{(\alpha_j)_{j\in\mathbb{N}}} = \Big\{ \bigcap_{j=1}^{\infty} \pi_{\alpha_j}^{-1}[E_{\alpha_j}] : E_{\alpha_1} \in \mathcal{A}_{\alpha_1}, E_{\alpha_2} \in \mathcal{A}_{\alpha_2}, \dots \Big\}.$$

- It is easy to verify that $\bigcup_{(\alpha_j)_{j\in\mathbb{N}}\subseteq A} \sigma(\mathcal{F}_{(\alpha_j)_{j\in\mathbb{N}}})$ is a σ-algebra.

Arbitrary product measure spaces

Theorem (Infinite product measure theorem (IPMT))

Let $\{(X_\alpha, \mathcal{A}_\alpha, \mu_\alpha) : \alpha \in A\}$ be a family of probability spaces. Then there exists a product measure on the product space $(\prod_{\alpha\in A} X_\alpha, \bigotimes_{\alpha\in A} \mathcal{A}_\alpha)$.

Proof. By the previous discussion we define $\mu : \bigotimes_{\alpha\in A} \mathcal{A}_\alpha \to [0,1]$ by

$$\mu(A) = \bigotimes_{n\in\mathbb{N}} \mu_{\alpha_n}(A), \qquad \text{if} \qquad A \in \sigma(\mathcal{F}_{(\alpha_n)_{n\in\mathbb{N}}})$$

for some $(\alpha_n)_{n\in\mathbb{N}} \subseteq A$, since

$$\bigotimes_{\alpha\in A} \mathcal{A}_\alpha = \bigcup_{(\alpha_j)_{j\in\mathbb{N}}\subseteq A} \sigma(\mathcal{F}_{(\alpha_j)_{j\in\mathbb{N}}}).$$

By the previous theorem μ is well-defined measure. It will be denoted by $\bigotimes_{\alpha\in A} \mu_\alpha$ and called the product measure on $(\prod_{\alpha\in A} X_\alpha, \bigotimes_{\alpha\in A} \mathcal{A}_\alpha)$. $\square$

Approximating classes and compact classes

Definition (Compact classes)

A family $\mathcal{K}$ of subsets of a set X is called a compact class for X if for any $(K_n)_{n\in\mathbb{N}} \subseteq \mathcal{K}$ with $\bigcap_{n\in\mathbb{N}} K_n = \emptyset$, there is $m \in \mathbb{N}$ such that $\bigcap_{n=1}^{m} K_n = \emptyset$.

Theorem (Approximating compact classes)

Let μ be a nonnegative additive set function on an algebra $\mathcal{A}$. Suppose that there exists a compact class $\mathcal{K}$ approximating μ in the following sense: for every $A \in \mathcal{A}$ and every $\varepsilon > 0$, there exist $K_\varepsilon \in \mathcal{K}$ and $A_\varepsilon \in \mathcal{A}$ such that $A_\varepsilon \subseteq K_\varepsilon \subseteq A$ and $\mu(A \setminus A_\varepsilon) < \varepsilon$. Then μ is countably additive.

- *In particular, this is true if the compact class $\mathcal{K}$ is contained in $\mathcal{A}$ and for any $A \in \mathcal{A}$ one has the equality*

$$\mu(A) = \sup\{\mu(K) : K \subseteq A \text{ and } K \in \mathcal{K}\}.$$

Proof

Proof. Suppose that $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ is a decreasing sequence of sets such that $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, we have to show that $\lim_{n\to\infty} \mu(A_n) = 0$.

- Fix $\varepsilon > 0$, then for every $n \in \mathbb{N}$ there are $K_n \in \mathcal{K}$ and $B_n \in \mathcal{A}$ so that $B_n \subseteq K_n \subseteq A_n$ and $\mu(A_n \setminus B_n) < \varepsilon 2^{-n}$.
- It is clear that $\bigcap_{n\in\mathbb{N}} K_n \subseteq \bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Hence there is $m \in \mathbb{N}$ such that $\bigcap_{n=1}^m K_n = \emptyset$, then $\bigcap_{n=1}^m B_n = \emptyset$. Note that
$$A_m = \bigcap_{n=1}^m A_n \subseteq \bigcup_{n=1}^m (A_n \setminus B_n).$$
- Indeed, if $x \in A_m$ then $x \in A_n$ for all $n \leq m$. If $x \notin \bigcup_{n=1}^m (A_n \setminus B_n)$, then $x \notin A_n \setminus B_n$ for all $n \leq m$, which means that $x \in B_n$ for all $n \leq m$, hence $x \in \bigcap_{n=1}^m B_n = \emptyset$, which is impossible!
- Hence $\lim_{m\to\infty} \mu(A_m) = 0$, since
$$\mu(A_m) \leq \sum_{n=1}^m \mu(A_n \setminus B_n) \leq \sum_{n=1}^m \varepsilon 2^{-n} \leq \varepsilon. \qquad \square$$

Basic properties of compact classes

Proposition

Let $\mathcal{K}$ be a compact class of subsets of a set X. Then the class $\mathcal{K}_{<\sigma}$ of finite unions of elements from $\mathcal{K}$ is a compact class. Moreover, the class $\mathcal{K}_{<\sigma\delta}$ of countable intersections of finite unions of elements from $\mathcal{K}$ is a compact class as well.

Proof. It suffices to prove that $\mathcal{K}_{<\sigma}$ is a compact class. Let

$$A_i = \bigcup_{n=1}^{m_i} K_{i,n},$$

where $K_{i,n} \in \mathcal{K}$, be such that $\bigcap_{i=1}^{k} A_i \neq \emptyset$ for every $k \in \mathbb{N}$. Our aim is to show that

$$\bigcap_{i\in\mathbb{N}} A_i \neq \emptyset.$$

Proof

- Let $M = \prod_{i\in\mathbb{N}}([1, m_i] \cap \mathbb{Z})$. Let
$$M_k = \Big\{\nu = (\nu_i)_{i\in\mathbb{N}} \in M : \bigcap_{n=1}^{k} K_{i,\nu_i} \neq \emptyset\Big\}.$$
- Note that $M_k \neq \emptyset$ for all $k \in \mathbb{N}$, which follows from the relation
$$\bigcup_{\nu\in M} \bigcap_{i=1}^{k} K_{i,\nu_i} = \bigcap_{i=1}^{k} A_i \neq \emptyset.$$
- Moreover, $M_{k+1} \subseteq M_k$ for all $k \in \mathbb{N}$. We prove that $\bigcap_{k\in\mathbb{N}} M_k \neq \emptyset$.
- Then taking $\nu \in \bigcap_{k\in\mathbb{N}} M_k$ we see that $\nu \in M_k$ for all $k \in \mathbb{N}$, hence $\bigcap_{n=1}^{k} K_{i,\nu_i} \neq \emptyset$, and consequently $\bigcap_{n=1}^{\infty} K_{i,\nu_i} \neq \emptyset$ by the compactness of $\mathcal{K}$. This implies that $\bigcap_{i\in\mathbb{N}} A_i \neq \emptyset$, since
$$\bigcap_{n=1}^{\infty} K_{i,\nu_i} \subseteq \bigcup_{\nu\in M} \bigcap_{i=1}^{k} K_{i,\nu_i} = \bigcap_{i=1}^{k} A_i \neq \emptyset \quad \text{for all} \quad k \in \mathbb{N}.$$

Proof

- To prove that $\bigcap_{k\in\mathbb{N}} M_k \neq \emptyset$, we pick

$$\nu^{(k)} = (\nu_n^{(k)})_{n\in\mathbb{N}} \in M_k \quad \text{for every} \quad k \in \mathbb{N}.$$

- Since $1 \leq \nu_n^{(k)} \leq m_n$ for all $n, k \in \mathbb{N}$, then there exist infinitely many indices $k \in \mathbb{N}$ such that the numbers $\nu_1^{(k)} = \mu_1$ for some $\mu_1 \in [1, m_1]$.
- By induction, we construct a sequence $\mu = (\mu_i)_{i\in\mathbb{N}} \in M$ such that, for every $n \in \mathbb{N}$, there exist infinitely many $k \in \mathbb{N}$ with the property that

$$\nu_i^{(k)} = \mu_i \quad \text{for all} \quad i \in \{1, \ldots, n\}. \qquad (*)$$

- We show that

$$\mu \in \bigcap_{k\in\mathbb{N}} M_k.$$

  We fix an arbitrary $n \in \mathbb{N}$ and pick an integer $k > n$ satisfying (*), then $\nu^{(k)} = (\mu_1, \ldots, \mu_n, \nu_{n+1}^{(k)}, \nu_{n+2}^{(k)}, \ldots) \in M_k \subseteq M_n$. This means that $\mu = (\mu_1, \ldots, \mu_n, \mu_{n+1}, \mu_{n+2}, \ldots) \in M_n$, since the membership in M_n is determined by the first n coordinates of a sequence. $\square$

Compact classes in products

Lemma

Let $\{(X_\alpha, \mathcal{A}_\alpha) : \alpha \in A\}$ be a family of measurable spaces. Suppose that for every $\alpha \in A$ we are given a compact class $\mathcal{K}_\alpha$ for X_α. Then, the class $\mathcal{C}_{<\sigma\delta}$ of countable intersections of finite unions of cylindrical sets from

$$\mathcal{C} = \Big\{ \Big(\prod_{\alpha \in F} K_\alpha \Big) \times \Big(\prod_{\beta \in A \setminus F} X_\beta \Big) : K_\alpha \in \mathcal{K}_\alpha,\ \alpha \in F \subseteq A,\ |F| < \infty \Big\}$$

is compact as well.

Proof. By the previous proposition it suffices to verify the compactness for

$$\mathcal{C}_0 = \Big\{ K_\alpha \times \prod_{\beta \in A \setminus \{\alpha\}} X_\beta : K_\alpha \in \mathcal{K}_\alpha,\ \alpha \in A \Big\},$$

since $\mathcal{C} = \{C_1 \cap \ldots \cap C_n : C_1, \ldots, C_n \in \mathcal{C}_0, n \in \mathbb{N}\}$.

Proof

- Let $(C_i)_{i\in\mathbb{N}} \subseteq \mathcal{C}_0$ be a family of cylinders such that
$$C_i = K_{\alpha_i}^{(i)} \times \prod_{\beta\in A\setminus\{\alpha_i\}} X_\beta \in \mathcal{C}_0$$
with $K_{\alpha_i}^{(i)} \in \mathcal{K}_{\alpha_i}$.
- Setting $S = \{\alpha_i : i \in \mathbb{N}\}$, note that
$$\bigcap_{i\in\mathbb{N}} C_i = \Big(\prod_{\alpha\in S} Q_\alpha\Big) \times \Big(\prod_{\beta\in S^c} X_\alpha\Big), \quad \text{where} \quad Q_\alpha = \bigcap_{\substack{i\in\mathbb{N}\\ \alpha_i=\alpha}} K_{\alpha_i}^{(i)}.$$
- If $\bigcap_{i\in\mathbb{N}} C_i = \emptyset$, then so one of the sets $Q_\alpha = \emptyset$. By the compactness of the class $\mathcal{K}_\alpha$, there exists $n \in \mathbb{N}$ such that $K_\alpha^{(1)} \cap \ldots \cap K_\alpha^{(n)} = \emptyset$.
- Then we have
$$C_1 \cap \ldots \cap C_n = \emptyset. \qquad \square$$

Kolmogorov extension theorem

- In many problems of measure theory and probability theory and their applications, one has to construct measures on products of measurable spaces that are more complicated than product measures.
- Now we prove the principal result in this direction: the Kolmogorov extension theorem on consistent probability distributions.
- The original Kolmogorov's result was concerned with measures on products of real lines. We present an abstract formulation that goes back to E. Marczewski.
- Let $T \neq \emptyset$. Suppose that a family $\{(\Omega_t, \mathcal{B}_t) : t \in T\}$ of measurable spaces is given. For every nonempty set $\Lambda \subset T$, we denote by

$$\Omega_\Lambda = \prod_{t \in \Lambda} \Omega_t, \quad \text{and} \quad \mathcal{B}_\Lambda = \bigotimes_{t \in \Lambda} \mathcal{B}_t,$$

the corresponding product space and product σ-algebra, respectively.

Kolmogorov extension theorem with Marczewski's proof

Theorem (Kolmogorov extension theorem **(KET)**)

Let $T \neq \emptyset$ and let $\{(\Omega_t, \mathcal{B}_t, \mu_t) : t \in T\}$ be a family of probability spaces.

- *Suppose that for every finite sets $\Lambda_1 \subseteq \Lambda_2$, the image of the measure μ_{Λ_2} under the projection from Ω_{Λ_2} to Ω_{Λ_1} coincides with μ_{Λ_1}, i.e.*

$$\mu_{\Lambda_2}\Big(E \times \prod_{t \in \Lambda_2 \setminus \Lambda_1} \Omega_t\Big) = \mu_{\Lambda_1}(E) \quad \text{for every} \quad E \in \mathcal{B}_{\Lambda_1}.$$

- *Moreover, suppose that for every $t \in T$, the measure μ_t on $\mathcal{B}_t$ has an approximating compact class $\mathcal{K}_t \subseteq \mathcal{B}_t$.*

Then, there exists a probability measure μ on $(\prod_{t \in T} \Omega_t, \bigotimes_{t \in T} \mathcal{B}_t)$ such that the image of μ under the natural projection from Ω to Ω_Λ is μ_Λ for each finite set $\Lambda \subset T$, i.e.

$$\mu\Big(E \times \prod_{t \in T \setminus \Lambda} \Omega_t\Big) = \mu_\Lambda(E) \quad \text{for every} \quad E \in \mathcal{B}_\Lambda.$$

Proof

Proof. We construct a probability measure μ on $(\prod_{t\in T}\Omega_t, \bigotimes_{t\in T}\mathcal{B}_t)$, which is called the projective limit of the measures μ_Λ for finite $\Lambda \subseteq T$.

- Every set $B \in \mathcal{B}_\Omega$ can be identified with the cylindrical set $C_\Lambda = B \times \prod_{t\in T\setminus\Lambda}\Omega_t$. It is clear that the set

$$\mathcal{A} = \Big\{B \times \prod_{t\in T\setminus\Lambda}\Omega_t : B \in \mathcal{B}_\Lambda,\ \Lambda \subseteq T,\ |\Lambda| < \infty\Big\},$$

  forms an algebra generated by the semi-algebra of finite products

$$\mathcal{S} = \Big\{\Big(\prod_{\alpha\in\Lambda} B_\alpha\Big) \times \Big(\prod_{t\in A\setminus\Lambda}\Omega_t\Big) : B_\alpha \in \mathcal{B}_\alpha,\ \alpha \in \Lambda \subseteq T,\ |\Lambda| < \infty\Big\}.$$

- On the algebra $\mathcal{A}$, we define the set function by

$$\mu(C_\Lambda) = \mu_\Lambda(B), \quad \text{whenever} \quad C_\Lambda = B \times \prod_{t\in T\setminus\Lambda}\Omega_t.$$

Proof

- The consistency condition yields that μ is well-defined, i.e. $\mu(C_\Lambda)$ is independent of the representation of C_Λ. Indeed, suppose that
$$C_\Lambda = B \times \prod_{t \in T \setminus \Lambda} \Omega_t = B' \times \prod_{t \in T \setminus \Lambda'} \Omega_t = C_{\Lambda'}$$
for some finite $\Lambda \subseteq \Lambda'$ and $B \in \mathcal{B}_\Lambda$ and $B' \in \mathcal{B}_{\Lambda'}$. This implies that
$$B' = B \times \prod_{t \in \Lambda' \setminus \Lambda} \Omega_t,$$
hence by the consistency condition we obtain that
$$\mu(C_{\Lambda'}) = \mu_{\Lambda'}(B') = \mu_{\Lambda'}\Big(B \times \prod_{t \in \Lambda' \setminus \Lambda} \Omega_t\Big) = \mu_\Lambda(B) = \mu(C_\Lambda).$$
- We verify the countable additivity of the set function μ on the algebra $\mathcal{A}$. Note that every cylindrical set from $\mathcal{A}$ can be approximated from inside by countable intersections of finite unions of products from $\mathcal{S}$.

Proof

- By the previous lemma the class $\mathcal{C}_{<\sigma}$ of finite unions of cylindrical sets from
$$\mathcal{C} = \left\{ \Big(\prod_{\alpha\in\Lambda} K_\alpha \Big) \times \Big(\prod_{t\in T\setminus\Lambda} \Omega_t \Big) : K_\alpha \in \mathcal{K}_\alpha,\ \alpha \in \Lambda \subseteq A,\ |\Lambda| < \infty \right\}$$
is compact. We prove that μ on the algebra $\mathcal{A}$ is approximated by the compact class $\mathcal{C}_{<\sigma}$.
- We show that for every $B = (\prod_{\alpha\in\Lambda} B_\alpha) \times (\prod_{t\in T\setminus\Lambda} \Omega_\beta) \in \mathcal{S}$, and for every $\varepsilon > 0$ there exists $K = (\prod_{\alpha\in\Lambda} K_\alpha) \times (\prod_{t\in T\setminus\Lambda} \Omega_\beta) \in \mathcal{C}$ so that
$$\mu(B \setminus K) < \varepsilon.$$
- For every $\alpha \in \Lambda$ it suffices to find $K_\alpha \in \mathcal{K}_\alpha$ such that $K_\alpha \subseteq B_\alpha$ and $\mu_\alpha(B_\alpha \setminus K_\alpha) < \varepsilon n^{-1}$. Then we see that
$$B\setminus K \subset \bigcup_{\alpha\in\Lambda} \Big((B_\alpha \setminus K_\alpha) \times \prod_{t\in\Lambda\setminus\{\alpha\}} \Omega_t \Big).$$

Proof

- This implies that

$$\mu(B\backslash K) \leq \sum_{\alpha\in\Lambda} \mu_\Lambda((B_\alpha\backslash K_\alpha) \times \prod_{t\in\Lambda\backslash\{\alpha\}} \Omega_t) = \sum_{\alpha\in\Lambda} \mu_\alpha(B_\alpha\backslash K_\alpha) < \varepsilon.$$

which completes the proof. □

Remarks

- It is clear that the Kolmogorov extension theorem is applicable if μ_Λ are consistent Radon measures.
- There is a very interesting construction (due to Andersen–Jessen) of a consistent family of measures that has no Kolmogorov extension.
- In particular, the Andersen–Jessen example shows that in **(KET)**, in contrast to **(IPMT)**, the assumption that the measures μ_t on $\mathcal{B}_t$ have approximating compact classes $\mathcal{K}_t \subseteq \mathcal{B}_t$ cannot be omitted.

Further remarks

Remarks

- Let $\{(X_n, \mathrm{Bor}(X_n), \mu_n) : n \in \mathbb{N}\}$ be a family of probability Borel spaces, where $(X_n)_{n\in\mathbb{N}}$ is a sequence of second countable metric spaces. If $X = \prod_{n\in\mathbb{N}} X_n$, then
$$\mathrm{Bor}(X) = \bigotimes_{n\in\mathbb{N}} \mathrm{Bor}(X_n),$$
and the product measure $\bigotimes_{n\in\mathbb{N}} \mu_n$ is in fact defined on $\mathrm{Bor}(X)$.
- In general case, if $\{(X_\alpha, \mathrm{Bor}(X_\alpha), \mu_\alpha) : \alpha \in A\}$ is a family of probability Borel spaces and $X = \prod_{\alpha\in A} X_\alpha$, the best what can be said is that
$$\bigotimes_{\alpha\in A} \mathrm{Bor}(X_\alpha) \subseteq \mathrm{Bor}(X),$$
and in fact $\mathrm{Bor}(X)$ may be much larger than the left hand side.

Example

Example

- Let $X = \mathbb{R}^{[0,\infty)} = \prod_{t\in[0,\infty)} X_t$, where $X_t = \mathbb{R}$ for all $t \in [0,\infty)$. In this case $\mathrm{Bor}(X)$ is strictly larger than the product σ-algebra, and

$$\bigotimes_{t\in[0,\infty)} \mathrm{Bor}(X_t) \subset \mathrm{Bor}(X).$$

- One can easily show that $C([0,\infty),\mathbb{R})$, the space of all real-valued continuous functions on $[0,\infty)$, is an $\mathcal{F}_{\sigma\delta}$ set, thus

$$C([0,\infty),\mathbb{R}) \in \mathrm{Bor}(X). \qquad \text{Justify this!}$$

  But on the other hand

$$C([0,\infty),\mathbb{R}) \notin \bigotimes_{t\in[0,\infty)} \mathrm{Bor}(X_t). \qquad \text{Justify this!}$$