Lecture 3

Set functions, measures and their properties

Graduate Real Analysis

Fall Semester

Finitely and countably additive set functions

Finitely and countably additive set functions

Let X be a set and let $\mathcal{U} \subseteq \mathcal{P}(X)$ be such that $\emptyset \in \mathcal{U}$. Let $\nu : \mathcal{U} \to [0, \infty]$ be a set function such that $\nu(\emptyset) = 0$. We say that the set function

(a) ν is **finitely additive** on $\mathcal{U}$ if for any disjoint finite collection of sets $A_1, \ldots, A_n \in \mathcal{U}$ such that $\bigcup_{j=1}^n A_j \in \mathcal{U}$ we have

$$\nu\Big(\bigcup_{j=1}^{n} A_j\Big) = \sum_{j=1}^{n} \nu(A_j).$$

(b) ν is **countably additive** on $\mathcal{U}$ if for any disjoint countable collection of sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\bigcup_{j=1}^\infty A_j \in \mathcal{U}$ we have

$$\nu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \nu(A_j).$$

Finitely and countably subadditive set functions

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(a) ν is **finitely subadditive** on $\mathcal{U}$ if for any finite collection of sets $A_1, \ldots, A_n \in \mathcal{U}$ such that $\bigcup_{j=1}^n A_j \in \mathcal{U}$ we have

$$\nu\Big(\bigcup_{j=1}^{n} A_j\Big) \leq \sum_{j=1}^{n} \nu(A_j).$$

(b) ν is **countably subadditive** on $\mathcal{U}$ if for any countable collection of sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\bigcup_{j=1}^\infty A_j \in \mathcal{U}$ we have

$$\nu\Big(\bigcup_{j=1}^{\infty} A_j\Big) \leq \sum_{j=1}^{\infty} \nu(A_j).$$

Some remarks

Remarks

Let X be a set and let $\mathcal{U} \subseteq \mathcal{P}(X)$ be such that $\emptyset \in \mathcal{U}$. Let $\nu : \mathcal{U} \to [0, \infty]$ be a set function such that $\nu(\emptyset) = 0$.

- Then it is easy to see that countable additivity implies finite additivity.
- Indeed, if $A_1, \ldots, A_n \in \mathcal{U}$ are disjoint such that $\bigcup_{j=1}^n A_j \in \mathcal{U}$ then one can take $A_j = \emptyset$ for all $j > n$ and note that

$$\nu(\bigcup_{j=1}^{n} A_j) = \nu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \nu(A_j) = \sum_{j=1}^{n} \nu(A_j),$$

  since $\nu(A_j) = \nu(\emptyset) = 0$ for all $j > n$.
- Similar argument shows that countable subadditivity implies finite subadditivity.

From finite to countable additivity

Problem 1

Let μ be a finitely additive set function on an algebra of sets $\mathcal{A}$ such that $\mu(A) < \infty$ for every $A \in \mathcal{A}$. Prove that the following conditions are equivalent:

(a) the function μ is countably additive,

(b) the function μ is continuous at zero in the following sense: if $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ is a decreasing sequence of sets ($A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$) with $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, then

$$\lim_{n\to\infty} \mu(A_n) = 0.$$

From countable subadditivity to countable additivity

Problem 2

Let $\rho : \mathcal{E} \to [0, \infty]$ be a set function on a semi-algebra $\mathcal{E}$ of subsets of a set X and $\rho(\emptyset) = 0$. Suppose that ρ is finitely additive on $\mathcal{E}$ and let $\mathcal{A}$ be the algebra that consists of all finite disjoint unions of members of $\mathcal{E}$. Define a set function μ_0 on $\mathcal{A}$ by setting

$$\mu_0(A) = \sum_{i=1}^{n} \rho(E_i)$$

for $A = \bigcup_{j=1}^{n} E_j \in \mathcal{A}$, where $E_1, \ldots, E_n \in \mathcal{E}$ and $E_i \cap E_j = \emptyset$ if $i \neq j$.

(a) Prove that μ_0 is a well-defined additive set function on $\mathcal{A}$ such that $\mu_0(\emptyset) = 0$ and $\mu_0 = \rho$ on $\mathcal{E}$.

(b) If additionally ρ is countably subadditive on $\mathcal{E}$ then prove that μ_0 is countably additive on $\mathcal{A}$.

Volume function on $\mathbb{R}^d$

Recall from the previous lecture that the family of all rectangles is given by

$$J^d = J_o^d \cup J_{oc}^d \cup J_{co}^d \cup J_c^d,$$

where $J_o^d, J_{oc}^d, J_{co}^d, J_c^d$, are, respectively, the families of open, open-closed, closed-open, and closed rectangles in $\mathbb{R}^d$.

Volume function on $\mathbb{R}^d$

For $I \in J^d$ we define the **volume function** $\mathrm{v}_d : J^d \to [0, \infty]$ by

$$\mathrm{v}_d(I) = \mathrm{vol}_d(I) = \begin{cases} 0 & \text{if } I = \emptyset, \\ \prod_{j=1}^d (b_j - a_j) & \text{if } \mathrm{cl}(I) = \prod_{j=1}^d [a_j, b_j], \\ \infty & \text{otherwise.} \end{cases}$$

For $d = 1$ the volume function v_d coincides with the length function on $\mathbb{R}$.

The volume function is finitely additive

Definition

A union of rectangles is said to be **almost disjoint** if the interiors of the rectangles are disjoint.

Lemma

If $R = \bigcup_{j=1}^{n} R_j \in J^d$ *and* $R_1, \ldots, R_n \in J^d$ *are almost disjoint, then*

$$\mathrm{v}_d(R) = \sum_{j=1}^{n} \mathrm{v}_d(R_j).$$

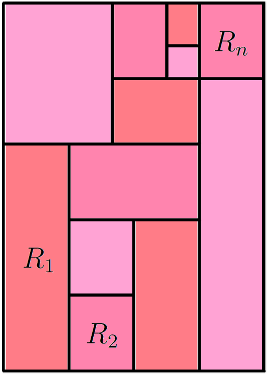
Proof. We consider the grid formed by extending indefinitely the sides of all rectangles $R_1, \ldots, R_n \in J^d$.

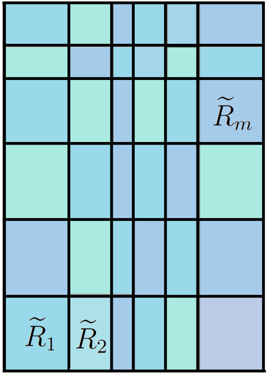
- This extension yields finitely many new rectangles $\widetilde{R}_1, \ldots, \widetilde{R}_m \in J^d$ and a partition $J_1, \ldots, J_n$ of the set $\{1, \ldots, m\}$ such that both unions

Proof

$$R = \bigcup_{j=1}^{m} \widetilde{R}_j \quad \text{and} \quad R_k = \bigcup_{j \in J_k} \widetilde{R}_j \quad \text{for} \quad k \in \{1, \dots, n\}$$

are almost disjoint.





Proof

- For the rectangle R, for example, we see that $\mathrm{v}_d(R) = \sum_{j=1}^{m} \mathrm{v}_d(\widetilde{R}_j)$, since the grid actually partitions the sides of R and each $\widetilde{R}_j$ consists of taking products of the intervals in these partitions.
- Thus when adding the volumes of the $\widetilde{R}_j$ we are summing the corresponding products of lengths of the intervals that arise. Since this also holds for the other rectangles $R_1, \ldots, R_n$ we conclude that

$$\mathrm{v}_d(R) = \sum_{j=1}^{m} \mathrm{v}_d(\widetilde{R}_j) = \sum_{k=1}^{n} \sum_{j \in J_k} \mathrm{v}_d(\widetilde{R}_j) = \sum_{k=1}^{n} \mathrm{v}_d(R_k). \qquad \square$$

Problem 3

If $R = \bigcup_{j=1}^{n} R_j \in J^d$ and $R_1, \ldots, R_n \in J^d$ are not necessarily disjoint, then

$$\mathrm{v}_d(R) \leq \sum_{j=1}^{n} \mathrm{v}_d(R_j).$$

Measurable spaces and measure spaces

Measurable space and sets

If X is a set and $\mathcal{A} \subseteq \mathcal{P}(X)$ is a σ-algebra, $(X, \mathcal{A})$ is called a **measurable space** and the sets in $\mathcal{A}$ are called **measurable sets**.

Definition of a measure

Let $(X, \mathcal{A})$ be a measurable space. A **measure** on $\mathcal{A}$ (or on $(X, \mathcal{A})$, or simply on X if $\mathcal{A}$ is understood) is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$,
2. if $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ is a sequence of disjoint sets in $\mathcal{A}$, then

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j).$$

Measurable space

If μ is a measure on $(X, \mathcal{A})$, then $(X, \mathcal{A}, \mu)$ is called a **measure space**.

Useful terminology

In other words, a measure $\mu : \mathcal{A} \to [0, \infty]$ on a σ-algebra $\mathcal{A}$ is a countably additive set function such that $\mu(\emptyset) = 0$.

Terminology

Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. If $\mu(X) < \infty$, μ is called **finite**. This immediately implies that $\mu(E) < \infty$ for all $E \in \mathcal{A}$, since $\mu(X) = \mu(E) + \mu(E^c)$, .
2. If $\mu(X) = 1$, μ is called **probability measure**.
3. If $X = \bigcup_{j \in \mathbb{N}} E_j$, where $E_j \in \mathcal{A}$ and $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$, μ is called σ-**finite**.
4. If for each $E \in \mathcal{A}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{A}$ with $F \subseteq E$ and $0 < \mu(F) < \infty$, μ is called **semifinite**.

Examples of measures

- Let $X \neq \emptyset$ and fix $x_0 \in X$. The **Dirac delta** measure on $(X, \mathcal{P}(X))$ is defined by
$$\delta_{x_0}(E) = \begin{cases} 0 & \text{if } x_0 \notin E, \\ 1 & \text{if } x_0 \in E, \end{cases} \qquad E \subseteq X.$$
- Let $X \neq \emptyset$ be countable. The **counting measure** on $(X, \mathcal{P}(X))$ is defined by
$$\mu(E) = \sum_{x \in X} \delta_x(E), \qquad E \subseteq X.$$
- Let X be an infinite set, and define a set function on $(X, \mathcal{P}(X))$ by
$$\mu(E) = \begin{cases} \infty & \text{if } E \text{ is infinite}, \\ 0 & \text{if } E \text{ is finite}, \end{cases} \qquad E \subseteq X.$$

  Then μ is a finitely additive set function, which is **not** a measure.

Examples of measures

- Let $X \neq \emptyset$ be a finite set. The **normalized counting measure** on $(X, \mathcal{P}(X))$ is defined by

$$\mu(E) = \frac{1}{\#X} \sum_{x \in X} \delta_x(E), \qquad E \subseteq X.$$

Then $\mu(X) = 1$, which means that μ is a probability measure on X.

Problem 4

Let X be an uncountable set, and let

$$\mathcal{A} = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}$$

be the algebra of countable or co-countable sets. Define

$$\mu(E) = \begin{cases} 1 & \text{if } E \text{ is uncountable}, \\ 0 & \text{otherwise}, \end{cases} \qquad E \in \mathcal{A}.$$

Show that μ is the measure on $(X, \mathcal{A})$.

Basic properties of measures

Theorem

Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) **(Monotonicity)** If $E, F \in \mathcal{A}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.

(b) **(Subadditivity)** If $(E_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$, then $\mu\big(\bigcup_{j=1}^{\infty} E_j\big) \leq \sum_{j=1}^{\infty} \mu(E_j)$.

(c) **(Continuity from below)** If $(E_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ and $E_1 \subseteq E_2 \subseteq \ldots$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

(d) **(Continuity from above)** If $(E_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ and $E_1 \supseteq E_2 \supseteq \ldots$ and $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof: 1/2

Proof of (a). If $E \subseteq F$, then

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E). \qquad \square$$

Proof of (b). Let $F_1 = E_1$ and $F_{k+1} = E_{k+1} \setminus \bigcup_{j=1}^{k} E_j$ for $k \in \mathbb{N}$. Then F_k's are disjoint and $\bigcup_{j=1}^{n} F_j = \bigcup_{j=1}^{n} E_j$ for all $n \in \mathbb{N}$. By **(a)** we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) \leq \sum_{n=1}^{\infty} \mu(E_n). \qquad \square$$

Proof of (c). Setting $E_0 = \emptyset$ we obtain

$$\mu\left(\bigcup_{n=0}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n \setminus E_{n-1}) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j \setminus E_{j-1}) = \lim_{n\to\infty} \mu(E_n). \qquad \square$$

Proof: 2/2

Proof of (d). Let $F_j = E_1 \setminus E_j$ for $j \in \mathbb{N}$. Then $F_1 \subseteq F_2 \subseteq \dots$ and $\mu(E_1) = \mu(E_j) + \mu(F_j)$, and

$$\bigcup_{j=1}^{\infty} F_j = E_1 \setminus \bigcap_{j=1}^{\infty} E_j.$$

Then by **(c)** we conclude

$$\mu(E_1) - \mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \mu\left(E_1 \setminus \bigcap_{j=1}^{\infty} E_j\right) = \mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \lim_{n\to\infty} \mu(F_n)$$
$$= \lim_{n\to\infty} (\mu(E_1) - \mu(E_n)).$$

Since $\mu(E_1) < \infty$ we may subtract it from both sides and we are done. $\square$

Problem 5

The assumptions $\mu(E_1) < \infty$ in **(d)** cannot be dropped.

Further properties of measures

Problem 6

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $(E_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}$ and $\mu(E_m \cap E_n) = 0$ for every $m \neq n$, prove that

$$\mu\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) = \sum_{j\in\mathbb{N}} \mu(E_j).$$

Problem 7

Let $(X, \mathcal{M})$ be a measurable space and $\mathcal{A}$ an algebra of sets such that $\sigma(\mathcal{A}) = \mathcal{M}$. Suppose μ_1 and μ_2 are finite measures on $(X, \mathcal{M})$ such that $\mu_1(A) = \mu_2(A)$ for every $A \in \mathcal{A}$. Show that

$$\mu_1(A) = \mu_2(A)$$

for every $A \in \mathcal{M}$.

Null sets $\equiv$ negligible sets $\equiv$ sets of measure zero

Null sets

If $(X, \mathcal{A}, \mu)$ is a measure space, a set $E \in \mathcal{A}$ such that $\mu(E) = 0$ is called a **null set**. By subadditivity any countable union of null sets is a null set.

Almost everywhere

If a statement about points $x \in X$ is true except for x in some null set, we say that it is true μ-**almost everywhere** (abbreviated μ-**a.e.**), or for μ-**almost every** x.

Remark

If $E \in \mathcal{A}$ and $\mu(E) = 0$ and $F \subseteq E$, then $\mu(F) = 0$ by monotonicity of μ provided that $F \in \mathcal{A}$. In general, it need not to be true that $F \in \mathcal{A}$.

Complete measure

A measure whose domain includes all subsets of null sets is called **complete**.

Completion of σ-algebras

Theorem

Suppose that $(X, \mathcal{A}, \mu)$ is a measure space. Let

$$\mathcal{N} = \{N \in \mathcal{A} : \mu(N) = 0\},$$

and

$$\overline{\mathcal{A}} = \{E \cup F : E \in \mathcal{A} \text{ and } F \subseteq N \text{ for some } N \in \mathcal{N}\}.$$

Then

(a) $\overline{\mathcal{A}}$ is a σ-algebra.

(b) There is a unique extension $\bar{\mu}$ of μ to a complete measure on $\overline{\mathcal{A}}$.

Proof of (a): Since $\mathcal{A}$ and $\mathcal{N}$ are closed under countable unions, so is $\overline{\mathcal{A}}$. We show that $(E \cup F)^c \in \overline{\mathcal{A}}$, if $E \cup F \in \overline{\mathcal{A}}$, where $E \in \mathcal{A}$ and $F \subseteq N \in \mathcal{N}$. This will show that $\overline{\mathcal{A}}$ is a σ-algebra and the proof of (a) will be completed.

Proof: 1/2

We can assume that $E \cap N = \emptyset$ (otherwise we replace F and N by $F \setminus E$ and $N \setminus E$). Then $E \cup F = (E \cup N) \cap (N^c \cup F)$, so

$$(E \cup F)^c = (E \cup N)^c \cup (N \setminus F).$$

But $(E \cup N)^c \in \mathcal{A}$ and $N \setminus F \subseteq N$, so that $(E \cup F)^c \in \overline{\mathcal{A}}$. So $\overline{\mathcal{A}}$ is a σ-algebra as desired. $\square$

Proof of (b): (Construction of $\bar{\mu}$). For $E \cup F \in \overline{\mathcal{A}}$, we set

$$\bar{\mu}(E \cup F) = \mu(E).$$

This new measure $\bar{\mu}$ is well defined. Indeed, if $E_1 \cup F_1 = E_2 \cup F_2$, where $F_j \subseteq N_j \in \mathcal{N}$ for $i = 1, 2$, then $E_1 \subseteq E_2 \cup N_2$ and so

$$\mu(E_1) \leq \mu(E_2) + \mu(N_2) = \mu(E_2),$$

and likewise $\mu(E_2) \leq \mu(E_1)$. Thus $\bar{\mu}(E_1 \cup F_1) = \bar{\mu}(E_2 \cup F_2)$ as claimed. $\square$

Proof: 2/2

Proof of (b): (Completeness of $\bar{\mu}$). If $E \in \overline{\mathcal{A}}$ and $\bar{\mu}(E) = 0$ then we show that for any $E_0 \subseteq E$ we have $\bar{\mu}(E_0) = 0$ and $E_0 \in \overline{\mathcal{A}}$. Let $E = A \cup C$, where $A \in \mathcal{A}$ and $C \subseteq N \in \mathcal{N}$ and $\mu(N) = 0$. Note that

$$0 = \bar{\mu}(E) = \mu(A).$$

Thus $A \in \mathcal{N}$ and consequently $E_0 \subseteq E = A \cup C \subseteq A \cup N \in \mathcal{N}$, which yields that $E_0 \in \overline{\mathcal{A}}$ and $\bar{\mu}(E_0) = \mu(\emptyset) = 0$. □

Proof of (b): (Uniqueness of $\bar{\mu}$). Let $(X, \mathcal{M}, \nu)$ be a complete measure space such that $\mathcal{M} \supseteq \overline{\mathcal{A}}$ and $\nu(E) = \mu(E)$ for all $E \in \mathcal{A}$, then we show that $\nu = \bar{\mu}$ on $\overline{\mathcal{A}}$. Indeed, suppose that $E \in \overline{\mathcal{A}}$, where $E = A \cup F$ and $F \subseteq N \in \mathcal{N}$ and $\mu(N) = 0$. We may write $E = A \cup F = A \cup (F \setminus A)$, and $F \setminus A \subseteq F \subseteq N \in \mathcal{N}$. Hence

$$\bar{\mu}(E) = \bar{\mu}(A \cup (F \setminus A)) = \mu(A) = \nu(A) = \nu(A) + \nu(F \setminus A) = \nu(E),$$

since $F \setminus A \in \mathcal{M}$ and $\nu(F \setminus A) = 0$. Thus $\nu = \bar{\mu}$ on $\overline{\mathcal{A}}$. □

Nonatomic measure spaces

Nonatomic measure space

We say that a measure space $(X, \mathcal{M}, \mu)$ is **nonatomic** if for every $A \in \mathcal{M}$ with $\mu(A) > 0$ there exists $B \in \mathcal{M}$ such that

$$B \subseteq A \quad \text{and} \quad 0 < \mu(B) < \mu(A).$$

Problem 8

Let $(X, \mathcal{M}, \mu)$ be a nonatomic finite measure space. Show that for every $\varepsilon > 0$ and every $A \in \mathcal{M}$ with $\mu(A) > 0$ there exists $B \in \mathcal{M}$ such that

$$B \subseteq A \quad \text{and} \quad 0 < \mu(B) < \varepsilon.$$

Nonatomic measure spaces attain all intermediate values

Problem 9

Let $(X, \mathcal{M}, \mu)$ be a nonatomic finite measure space. Show that for every $A \in \mathcal{M}$ and every $\theta \in [0, \mu(A)]$ there exists $B \in \mathcal{M}$ such that

$$B \subseteq A \quad \text{and} \quad \mu(B) = \theta.$$

Hint

Inductively define the collections $\mathcal{H}_n$, the numbers h_n and the sets H_n as follows. Set $H_0 = \emptyset$ and $\mathcal{H}_0 = \{\emptyset\}$, and for $n \in \mathbb{N}$ let

$$\mathcal{H}_n = \left\{ H \in \mathcal{M} \colon H \subseteq A \setminus \bigcup_{k=0}^{n-1} H_k \ \text{ and } \ \mu\Big(\bigcup_{k=0}^{n-1} H_k\Big) + \mu(H) \leq \theta \right\},$$

$$h_n = \sup\{\mu(H) \colon H \in \mathcal{H}_n\},$$

and choose $H_n \in \mathcal{H}_n$ such that $\mu(H_n) \geq h_n - \min\left\{h_n, \frac{1}{n}\right\}$. Finally, consider the set $\bigcup_{n\in\mathbb{N}} H_n$.

Regularity of finite Borel measures

Borel measures

Let X be a topological space. A measure μ defined on a Borel σ-algebra $\mathrm{Bor}(X)$ is called a **Borel measure**. A pair $(X, \mathrm{Bor}(X))$ is called a **Borel space**, whereas a triple $(X, \mathrm{Bor}(X), \mu)$ is called a **Borel measure space**.

Problem 10

Let X be a metric space and let $(X, \mathrm{Bor}(X), \mu)$ be a finite Borel measure space. Show that

$$\mathrm{Bor}(X) = \mathcal{M},$$

where

$$\mathcal{M} = \Big\{ B \in \mathrm{Bor}(X) : \mu(B) = \sup\{\mu(F) : F \subseteq B \text{ and } F \text{ closed}\} = \inf\{\mu(U) : U \supseteq B \text{ and } U \text{ open}\} \Big\}.$$