Lecture 4

Outer measures, Caratheodory's theorem and Lebesgue measure on $\mathbb{R}^d$

Graduate Real Analysis

Fall Semester

Outer measures

Definition of an outer measure

An **outer measure** on a set $X \neq \emptyset$ is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

(i) $\mu^*(\emptyset) = 0$,

(ii) $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$,

(iii) $\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$.

- In other words, an outer measure $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is a monotone and countably subadditive set function such that $\mu^*(\emptyset) = 0$.
- The most common way to obtain outer measures is to start with a family $\mathcal{E}$ of "elementary sets" on which a notion of measure is defined (such as rectangles in the plane) and then to approximate arbitrary sets "from the outside" by countable unions of members of $\mathcal{E}$.

Outer measures from arbitrary set function

Proposition

Let $\mathcal{E} \subseteq \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be a set function such that $\emptyset, X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. For any $A \subseteq X$, define

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) : (E_j)_{j \in \mathbb{N}} \subseteq \mathcal{E} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.$$

Then μ^* is an outer measure.

Proof. For any $A \subseteq X$ there exists $(E_j)_{j=1}^{\infty}$ such that $A \subseteq \bigcup_{j=1}^{\infty} E_j$ (take $E_j = X$ for all $j \in \mathbb{N}$) so the definition of μ^* makes sense.

- Obviously $\mu^*(\emptyset) = 0$ (take $E_j = \emptyset$ for all $j \in \mathbb{N}$).
- We also have $\mu^*(A) \leq \mu^*(B)$ for any $A \subseteq B$. Indeed, if $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{E}$ and $B \subseteq \bigcup_{j=1}^{\infty} E_j$ then $A \subseteq \bigcup_{j=1}^{\infty} E_j$, so a covering $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{E}$ for B is also a covering for A. Hence $\mu^*(A) \leq \mu^*(B)$.

Proof:

- To prove the countable subadditivity, let $(A_j)_{j=1}^{\infty} \subseteq \mathcal{P}(X)$ and $\varepsilon > 0$. For each A_j there is $(E_{j,k})_{k=1}^{\infty}$ such that

$$A_j \subseteq \bigcup_{k=1}^{\infty} E_{j,k} \quad \text{and} \quad \sum_{k=1}^{\infty} \rho(E_{j,k}) \leq \mu^*(A_j) + \varepsilon 2^{-j}.$$

- But then if $A = \bigcup_{j=1}^{\infty} A_j$ we have $A \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} E_{j,k}$ and

$$\mu^*(A) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_{j,k}). \qquad \text{Why? Justify this!}$$

- Consequently

$$\mu^*(A) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_{j,k}) \leq \sum_{j=1}^{\infty} \left(\mu^*(A_j) + \varepsilon 2^{-j}\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon.$$

Since ε is arbitrary, we are done. $\square$

Volume function on $\mathbb{R}^d$

Recall from the previous lecture that the family of all rectangles is given by

$$J^d = J_o^d \cup J_{oc}^d \cup J_{co}^d \cup J_c^d,$$

where $J_o^d, J_{oc}^d, J_{co}^d, J_c^d$, are, respectively, the families of open, open-closed, closed-open, and closed rectangles in $\mathbb{R}^d$.

Volume function on $\mathbb{R}^d$

For $I \in J^d$ we define the **volume function** $\mathrm{v}_d : J^d \to [0, \infty]$ by

$$\mathrm{v}_d(I) = \mathrm{vol}_d(I) = \begin{cases} 0 & \text{if } I = \emptyset, \\ \prod_{j=1}^d (b_j - a_j) & \text{if } \mathrm{cl}(I) = \prod_{j=1}^d [a_j, b_j], \\ \infty & \text{otherwise.} \end{cases}$$

For $d = 1$ the volume function v_d coincides with the length function on $\mathbb{R}$.

Some outer measures induced by the volume

Problem 1

For every $E \subseteq \mathbb{R}^d$ we define

$$\lambda^*_{d,o}(E) = \inf\left\{ \sum_{n\in\mathbb{N}} \mathrm{v}_d(I_n) : (I_n)_{n\in\mathbb{N}} \subseteq J^d_o \text{ and } E \subseteq \bigcup_{n\in\mathbb{N}} I_n \right\},$$

$$\lambda^*_{d,oc}(E) = \inf\left\{ \sum_{n\in\mathbb{N}} \mathrm{v}_d(I_n) : (I_n)_{n\in\mathbb{N}} \subseteq J^d_{oc} \text{ and } E \subseteq \bigcup_{n\in\mathbb{N}} I_n \right\},$$

$$\lambda^*_{d,co}(E) = \inf\left\{ \sum_{n\in\mathbb{N}} \mathrm{v}_d(I_n) : (I_n)_{n\in\mathbb{N}} \subseteq J^d_{co} \text{ and } E \subseteq \bigcup_{n\in\mathbb{N}} I_n \right\},$$

$$\lambda^*_{d,c}(E) = \inf\left\{ \sum_{n\in\mathbb{N}} \mathrm{v}_d(I_n) : (I_n)_{n\in\mathbb{N}} \subseteq J^d_c \text{ and } E \subseteq \bigcup_{n\in\mathbb{N}} I_n \right\}.$$

Prove that each of these set functions is an outer measure on $\mathbb{R}^d$ and for any $E \subseteq \mathbb{R}^d$ we have $\lambda^*_{d,o}(E) = \lambda^*_{d,co}(E) = \lambda^*_{d,oc}(E) = \lambda^*_{d,c}(E)$.

d-dimensional Lebesgue measure

Definition

Let $\mathrm{v}_d : J^d \to [0, \infty]$ be the volume function on $\mathbb{R}^d$. The d-dimensional **Lebesgue outer measure** is defined for any $E \subseteq \mathbb{R}^d$ by setting

$$\lambda_d^*(E) = \inf \left\{ \sum_{n \in \mathbb{N}} \mathrm{v}_d(E_n) : (E_n)_{n \in \mathbb{N}} \subseteq J_{co}^d \quad \text{and} \quad E \subseteq \bigcup_{n \in \mathbb{N}} E_n \right\}.$$

- In other words, we have $\lambda_d^*(E) = \lambda_{d,co}^*(E)$ for any $E \subseteq \mathbb{R}^d$.
- If $d = 1$ we will abbreviate λ_d^* to λ^*.

Problem 2

For every rectangle $E \in J^d = J_o^d \cup J_{oc}^d \cup J_{co}^d \cup J_c^d$ prove that

$$\mathrm{v}_d(E) = \lambda_{d,o}^*(E) = \lambda_{d,co}^*(E) = \lambda_{d,oc}^*(E) = \lambda_{d,c}^*(E)$$

μ^*-measurable sets and Carathéodory's condition

Carathéodory's condition

If μ^* is an outer measure on X, a set $A \subseteq X$ is called μ^*-**measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for all} \quad E \subseteq X.$$

Remark

- By subadditivity of μ^* we clearly have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

- To prove that $A \subseteq X$ is μ^*-measurable, it suffices to prove the reverse inequality

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

  for all $E \subseteq X$ such that $\mu^*(E) < \infty$.

Intuitions behind the Carathéodory condition

- Suppose that $A \subseteq X = (a,b) \times (c,d) \subset \mathbb{R}^2$ and let $\mathcal{C}$ be a family of open rectangles in X and ρ be the area function. Define

$$\mu^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \rho(C_j) : (C_j)_{j\in\mathbb{N}} \subseteq \mathcal{C} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} C_j \right\},$$

  which is an outer measure that approximates the area of a bounded set A from the outside by using rectangles.
- Since $\mu^*(X) < \infty$ we can define an **inner measure** by setting

$$\mu_*(A) = \mu^*(X) - \mu^*(A^c),$$

  which approximates the area of a bounded set A from the inside.
- If $\mu^*(A) = \mu_*(A)$ it is reasonable to think that their common value is the area of A, which leads to the measurability condition

$$\mu^*(A) = \mu_*(A) \quad \Longleftrightarrow \quad \mu^*(X) = \mu^*(X \cap A) + \mu^*(X \cap A^c).$$

Carathéodory's Theorem

Carathéodory's Theorem

Let μ^* be an outer measure on X, and let

$$\mathcal{M}(\mu^*) = \left\{A \subseteq X : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for all} \quad E \subseteq X\right\}.$$

be the collection of μ^*-measurable sets. Then

(i) $\mathcal{M}(\mu^*)$ is a σ-algebra, (**Carathéodory's σ-algebra** induced by μ^*).

(ii) $(X, \mathcal{M}(\mu^*), \mu^*)$ is a complete measure space.

Proof. Step 1. We first show that $\mathcal{M}(\mu^*)$ is an algebra.

- Observe that $\mathcal{M}(\mu^*)$ is closed under complements since the definition of μ^*-measurability of A is symmetric in A and A^c.
- Next, if $A, B \in \mathcal{M}(\mu^*)$, then we will show that $A \cup B \in \mathcal{M}(\mu^*)$.

For any $E \subseteq X$ we have to show that

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

Proof: 1/5

Note that $(A \cup B)^c = A^c \cap B^c$ and

$$A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B).$$

Therefore, by subadditivity of μ^* and Carathéodory's condition we obtain

$$\begin{aligned} &\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \\ \leq\ &\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ =\ &\mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E). \end{aligned}$$

Problem 3

Show that, in fact, we have equality and subadditivity of μ^* is not needed!

Finally, we conclude that

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \geq \mu^*(E).$$

Thus $A \cup B \in \mathcal{M}(\mu^*)$, which shows that $\mathcal{M}(\mu^*)$ is an algebra.

Proof: 2/5

Step 2. We show that $\mathcal{M}(\mu^*)$ is finitely additive.

- Let $A, B \in \mathcal{M}(\mu^*)$ be such that $A \cap B = \emptyset$. Then $B \subseteq A^c$ and

$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c) = \mu^*(A) + \mu^*(B).$$

Step 3. We now show that $\mathcal{M}(\mu^*)$ is a σ-algebra.

- Since $\mathcal{M}(\mu^*)$ is an algebra, it suffices to show that $\mathcal{M}(\mu^*)$ is closed under countable disjoint unions. Let $(A_j)_{j=1}^{\infty} \subseteq \mathcal{M}(\mu^*)$ be such that $A_i \cap A_j = \emptyset$ for $i \neq j$, and define

$$B_0 = \emptyset, \quad B_n = \bigcup_{j=1}^{n} A_j, \quad B = \bigcup_{j=1}^{\infty} A_j.$$

Then for any $E \subseteq X$ we have

$$\begin{aligned} \mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}). \end{aligned}$$

Proof: 3/5

Iterating $\mu^*(E \cap B_n) = \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1})$ we obtain

$$\mu^*(E \cap B_n) = \mu^*(E \cap A_n) + \mu^*(E \cap A_{n-1}) + \mu^*(E \cap B_{n-2}) = \sum_{j=0}^{n} \mu^*(E \cap A_j).$$

Therefore we see

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) = \sum_{j=0}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap B_n^c)$$

$$\geq \sum_{j=0}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap B^c),$$

since $B_n \subseteq B = \bigcup_{j=1}^{\infty} A_j$. Letting $n \to \infty$ we obtain

$$\mu^*(E) \geq \sum_{j=0}^{\infty} \mu^*(E \cap A_j) + \mu^*(E \cap B^c).$$

Proof: 4/5

Thus by subadditivity of μ^* we see that

$$\mu^*(E) \geq \sum_{j=0}^{\infty} \mu^*(E \cap A_j) + \mu^*(E \cap B^c) \geq \mu^*\left(\bigcup_{j=0}^{\infty}(E \cap A_j)\right) + \mu^*(E \cap B^c)$$
$$= \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E).$$

Step 4. We now show that μ^* is countably additive on $\mathcal{M}(\mu^*)$.
From the previous calculation we have

$$\mu^*(E) = \sum_{j=0}^{\infty} \mu^*(E \cap A_j) + \mu^*(E \cap B^c).$$

Taking $E = B = \bigcup_{j=1}^{\infty} A_j$ we obtain

$$\mu^*(B) = \sum_{j=1}^{\infty} \mu^*(A_j)$$

as claimed.

Proof: 5/5

Step 5. We finally show that μ^* is a complete measure on $\mathcal{M}(\mu^*)$.

Complete measures

Recall that a measure whose domain includes all subsets of null sets is called **complete**.

- Take any $A \subseteq X$ such that $\mu^*(A) = 0$. We will show $A \in \mathcal{M}(\mu^*)$. For any $E \subseteq X$ we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E \cap A^c) \leq \mu^*(E),$$

since $\mu^*(E \cap A) = 0$. Thus

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

and $A \in \mathcal{M}(\mu^*)$. The proof of Carathéodory's theorem is finished. $\square$

More about outer measures

Problem 4

Let $(X, \mathcal{M}, \mu)$ be a measure space. For every $E \subseteq X$ define

$$\mu^*(E) = \inf\{\mu(A) : A \supseteq E \text{ and } A \in \mathcal{M}\}.$$

(a) Prove that μ^* is an outer measure on X.

(b) Prove that $\mathcal{M} \subseteq \mathcal{M}(\mu^*)$.

(c) Prove that $\mu^*(E) = \mu(E)$ for any $E \in \mathcal{M}$.

Remark

The purpose of this problem is show that the measure space $(X, \mathcal{M}(\mu^*), \mu^*)$ may serve as a complete extension of $(X, \mathcal{M}, \mu)$.

d-dimensional Lebesgue measure

Definition

Let $\mathrm{v}_d : J^d \to [0, \infty]$ be the volume function on $\mathbb{R}^d$. The d-dimensional **Lebesgue outer measure** is defined for any $E \subseteq \mathbb{R}^d$ by setting

$$\lambda_d^*(E) = \inf\left\{ \sum_{n\in\mathbb{N}} \mathrm{v}_d(E_n) : (E_n)_{n\in\mathbb{N}} \subseteq J_{co}^d \quad \text{and} \quad E \subseteq \bigcup_{n\in\mathbb{N}} E_n \right\}.$$

- The d-dimensional **Lebesgue measure** λ_d is the restriction of λ_d^* to the Carathéodory σ-algebra $\mathcal{L}(\mathbb{R}^d) = \mathcal{M}(\lambda_d^*)$ of λ_d^*-measurable sets.
- $\mathcal{L}(\mathbb{R}^d) = \mathcal{M}(\lambda_d^*)$ will be called the **Lebesgue σ-algebra**.
- The members of $\mathcal{L}(\mathbb{R}^d)$ are the **Lebesgue measurable sets**.
- By Carathéodory's theorem $(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d), \lambda_d)$ is a complete measure space. For any rectangle $E \in J^d = J_o^d \cup J_{oc}^d \cup J_{co}^d \cup J_c^d$, we have $\lambda_d(E) = \mathrm{v}_d(E)$ and λ_d is σ-finite on $\mathbb{R}^d$.
- If $d = 1$ we shall abbreviate λ_d^* to λ^* and λ_d to λ.

Borel sets are Lebesgue measurable

Theorem

One has that $\mathrm{Bor}(\mathbb{R}^d) \subseteq \mathcal{L}(\mathbb{R}^d)$, since

$$\mathrm{Bor}(\mathbb{R}^d) = \sigma\Big(\Big\{ \bigcap_{i=1}^{d} H_{i,\xi_i} : \xi_1, \ldots, \xi_d \in \mathbb{Q}\Big\}\Big),$$

where $H_{i,\xi} = \mathbb{R}^{i-1} \times (-\infty, \xi) \times \mathbb{R}^{d-i}$ for $i \in \{1, \ldots, d\}$.

- The proof follows from the following proposition.

Proposition

For any $1 \in i \leq d$ and $\xi \in \mathbb{R}$ the half-space $H_{i,\xi}$ is Lebesgue measurable.

- For any $a = (a_1, \ldots, a_d)$, $b = (b_1, \ldots, b_d) \in [-\infty, \infty]^d$ we will write

$$[a, b) = \prod_{i=1}^{d} [a_i, b_i) \in J_{co}^d.$$

Proof: 1/2

We write $H = H_{i,\xi}$, and we recall that $\lambda_d^* = \lambda_{d,co}^*$.

- **Step 1.** We first show that

$$\mathrm{v}_d(I \cap H) + \mathrm{v}_d(I \setminus H) = \mathrm{v}_d(I) \quad \text{for every} \quad I \in J_{co}^d.$$

- If $I \subseteq H$ or $I \cap H = \emptyset$, this is obvious, since in the former case $I \cap H = I$ and $I \setminus H = \emptyset$, while in the latter case $I \cap H = \emptyset$ and $I \setminus H = I$.
- Otherwise $I = [a, b)$ with $a_i < \xi < b_i$, and then $I \cap H = [a, x)$ and $I \setminus H = [y, b)$, where

$$x = (b_1, \ldots, b_{i-1}, \xi, b_{i+1}, \ldots, b_d), \quad y = (a_1, \ldots, a_{i-1}, \xi, a_{i+1}, \ldots, a_d).$$

- In particular $I \cap H$, $I \setminus H \in J_{co}^d$ and we have

$$\begin{aligned} \mathrm{v}_d(I \cap H) + \mathrm{v}_d(I \setminus H) &= (\xi - a_i) \prod_{j \neq i} (b_j - a_j) + (b_i - \xi) \prod_{j \neq i} (b_j - a_j) \\ &= (b_i - a_i) \prod_{j \neq i} (b_j - a_j) = \mathrm{v}_d(I). \end{aligned}$$

Proof: 2/2

- **Step 2.** Let $A \subseteq \mathbb{R}^d$ be such that $\lambda_d^*(A) < \infty$ and let $\varepsilon > 0$.
- By the definition of λ_d^* there is a sequence $(I_j)_{j\in\mathbb{N}} \subseteq J_{co}^d$ such that

$$A \subseteq \bigcup_{j\in\mathbb{N}} I_j \quad \text{and} \quad \sum_{j\in\mathbb{N}} \mathrm{v}_d(I_j) \leq \lambda_d^*(A) + \varepsilon.$$

- By Step 1, we have $(I_j \cap H)_{j\in\mathbb{N}}, (I_j \setminus H)_{j\in\mathbb{N}} \subseteq J_{co}^d$ and

$$A \cap H \subseteq \bigcup_{j\in\mathbb{N}} (I_j \cap H), \quad A \setminus H \subseteq \bigcup_{j\in\mathbb{N}} (I_j \setminus H).$$

- Hence, by the definition of λ_d^* and by Step 1,

$$\begin{aligned} \lambda_d^*(A \cap H) + \lambda_d^*(A \setminus H) &\leq \sum_{j\in\mathbb{N}} \mathrm{v}_d(I_j \cap H) + \sum_{j\in\mathbb{N}} \mathrm{v}_d(I_j \setminus H) \\ &= \sum_{j\in\mathbb{N}} \mathrm{v}_d(I_j) \leq \lambda_d^*(A) + \varepsilon. \end{aligned}$$

- Since $\varepsilon > 0$ was arbitrary, $\lambda_d^*(A \cap H) + \lambda_d^*(A \cap H^c) \leq \lambda_d^*(A)$, and hence H is λ_d^*-measurable as desired. $\square$

d-dimensional Lebesgue measure — revised

Definition

Let $\mathrm{v}_d : J^d \to [0, \infty]$ be the volume function on $\mathbb{R}^d$. The d-dimensional **Lebesgue outer measure** is defined for any $E \subseteq \mathbb{R}^d$ by setting

$$\lambda_d^*(E) = \inf\left\{ \sum_{n\in\mathbb{N}} \mathrm{v}_d(E_n) : (E_n)_{n\in\mathbb{N}} \subseteq J_{co}^d \quad \text{and} \quad E \subseteq \bigcup_{n\in\mathbb{N}} E_n \right\}.$$

- The d-dimensional **Lebesgue measure** λ_d is the restriction of λ_d^* to the Carathéodory σ-algebra $\mathcal{L}(\mathbb{R}^d) = \mathcal{M}(\lambda_d^*)$ of λ_d^*-measurable sets.
- $\mathcal{L}(\mathbb{R}^d) = \mathcal{M}(\lambda_d^*)$ will be called the **Lebesgue σ-algebra**.
- The members of $\mathcal{L}(\mathbb{R}^d)$ are the **Lebesgue measurable sets**.
- By Carathéodory's theorem $(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d), \lambda_d)$ is a complete measure space. For any rectangle $E \in J^d = J_o^d \cup J_{oc}^d \cup J_{co}^d \cup J_c^d$, we have $\lambda_d(E) = \mathrm{v}_d(E)$ and λ_d is σ-finite on $\mathbb{R}^d$, and $\mathrm{Bor}(\mathbb{R}^d) \subseteq \mathcal{L}(\mathbb{R}^d)$.
- If $d = 1$ we shall abbreviate λ_d^* to λ^* and λ_d to λ.

Mertic outer measures

Definition

Let (X, ρ) be a metric space. An outer measure μ^* is called a **metric outer measure** if

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B) \quad \text{whenever} \quad \rho(A, B) > 0.$$

Problem 5

Let μ^* be an outer measure on a metric space (X, ρ). Prove that μ^* is a metric outer measure if and only if $\mathrm{Bor}(X) \subseteq \mathcal{M}(\mu^*)$.

Problem 6

Prove that the Lebesgue outer measure λ_d^* is a metric outer measure.

Borel measures and distribution functions

Suppose that μ is a finite Borel measure on $\mathbb{R}$ and let $F(x) = \mu((-\infty, x])$ be the **distribution function** of μ. Note that

- F is increasing, since μ is monotone.
- F is right continuous, since $\bigcap_{n\in\mathbb{N}}(-\infty, x_n] = (-\infty, x]$ and
$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \mu((-\infty, x_n]) = \mu((-\infty, x]) = F(x)$$
whenever $x_n \searrow x$ as $n \to \infty$.
- Moreover, if $-\infty < a < b \leq \infty$ then $(-\infty, b] = (-\infty, a] \cup (a, b]$ thus
$$\mu((a, b]) = F(b) - F(a),$$
where $(a, \infty] = (a, \infty)$ and $F(\infty) = \lim_{x\to\infty} F(x)$.

Our procedure will be to turn this process around and construct a measure μ starting from an increasing, right-continuous function F. As a particular case, if $F(x) = x$ we construct the Lebesgue measure on $\mathbb{R}$.

Lebesgue–Stieltjes outer measure

Problem 7

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right-continuous function. The **Lebesgue–Stieltjes outer measure** is defined for any $E \subseteq \mathbb{R}$ by

$$\mu_F^*(E) = \inf \left\{ \sum_{n\in\mathbb{N}} (F(b_n) - F(a_n)) : E \subseteq \bigcup_{n\in\mathbb{N}} (a_n, b_n] \right\}.$$

Prove that μ_F^* is an outer measure.

Definition

The previous problem and Carathéodory's theorem immediately yield:

- The **Lebesgue–Stieltjes measure** μ_F is the restriction of μ_F^* to the Carathéodory σ-algebra $\mathcal{L}_{\mu_F} = \mathcal{M}(\mu_F^*)$ of μ_F^*-measurable sets.
- $\mathcal{L}_{\mu_F} = \mathcal{M}(\mu_F^*)$ will be called the **Lebesgue–Stieltjes σ-algebra**.
- The members of $\mathcal{L}_{\mu_F}$ are the **Lebesgue–Stieltjes measurable sets**.
- By Carathéodory's theorem $(\mathbb{R}, \mathcal{L}_{\mu_F}, \mu_F)$ is a complete measure space.

Lebesgue–Stieltjes measure

Problem 8

Prove that μ_F^* is a metric outer measure and deduce $\mathrm{Bor}(\mathbb{R}) \subseteq \mathcal{L}_{\mu_F}(\mathbb{R})$.

Problem 9

Prove that $\mu_F((a,b]) = F(b) - F(a)$ for any $-\infty \leq a \leq b \leq \infty$.

Problem 10

Prove that the Lebesgue–Stieltjes measure μ_F is unique in the sense that if G is another such function, we have $\mu_F = \mu_G \iff F - G$ is constant. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$F(x) = \begin{cases} \mu((-\infty, x]) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu((x, 0]) & \text{if } x < 0, \end{cases}$$

then F is increasing and right continuous and $\mu = \mu_F$.

Hausdorff outer measure

Definition

Let (X, ρ) be a metric space, let $p \geq 0$ and $\delta > 0$. For every $A \subseteq X$ we define

$$H_{p,\delta}(A) = \inf \Big\{ \sum_{j \in \mathbb{N}} (\operatorname{diam} B_j)^p : A \subseteq \bigcup_{j \in \mathbb{N}} B_j \;\text{ and }\; \operatorname{diam} B_j \leq \delta \Big\},$$

with the convention that $\inf \emptyset = \infty$. The ***p*-dimensional Hausdorff outer measure** of A is

$$H_p(A) = \lim_{\delta \to 0^+} H_{p,\delta}(A).$$

- Here $\operatorname{diam} B = \sup\{\rho(x, y) : x, y \in B\}$, and we use the conventions $\operatorname{diam} \emptyset = 0$ and $(\operatorname{diam} B)^0 = 1$ for $B \neq \emptyset$.
- As δ decreases, the infimum is taken over a smaller family of coverings of A, so $H_{p,\delta}(A)$ increases and the above limit always exists in $[0, \infty]$.

Hausdorff outer measure

Problem 11

Let (X, ρ) be a metric space and let $p \geq 0$. Prove that

(a) $H_{p,\delta}$ is an outer measure on X for every $\delta > 0$;

(b) $H_{p,\delta}(A) \leq H_{p,\delta'}(A)$ for every $A \subseteq X$ and $0 < \delta' \leq \delta$, and consequently

$$H_p(A) = \lim_{\delta \to 0^+} H_{p,\delta}(A) = \sup_{\delta > 0} H_{p,\delta}(A);$$

(c) H_p is an outer measure on X.

Problem 12

Prove that the value of $H_{p,\delta}(A)$ does not change if the sets B_j are required to be closed, and also if they are required to be open. If $X = \mathbb{R}$, prove that one may equally well restrict the B_j to be closed intervals or open intervals.

Comments on the definition

- Hint to Problem 12: for closed sets it suffices to observe that $\operatorname{diam} B_j = \operatorname{diam} \operatorname{cl}(B_j)$; for open sets one replaces B_j by
$$U_j = \{x \in X : \rho(x, B_j) < \varepsilon 2^{-j-1}\},$$
whose diameter is at most $(\operatorname{diam} B_j) + \varepsilon 2^{-j}$.
- When p is an integer, the intuition behind the definition of H_p is that if $A \subseteq \mathbb{R}^d$ behaves like a p-dimensional set — for instance, if A is a relatively open subset of a p-dimensional linear subspace — then the portion of A lying in a region of diameter r should have p-dimensional size comparable to r^p.
- Requiring the covering sets to have arbitrarily small diameters is essential for H_p to capture the local geometry of irregular sets. Without this restriction, one could cover A simply by A itself, obtaining the crude bound
$$H_p(A) \leq (\text{diam } A)^p,$$
which generally does not reflect the true p-dimensional size of A.

Hausdorff measure

Problem 13

Prove that H_p is a metric outer measure on (X, ρ) and deduce from Problem 5 that $\mathrm{Bor}(X) \subseteq \mathcal{M}(H_p)$.

Definition

The restriction of H_p to $\mathrm{Bor}(X)$ is a measure, which we still denote by H_p and call the *p*-**dimensional Hausdorff measure**.

Problem 14

Prove that H_p is invariant under isometries of (X, ρ). Moreover, if Y is a set and $f, g : Y \to X$ satisfy

$$\rho(f(y), f(z)) \le C\rho(g(y), g(z)) \quad \text{for all} \quad y, z \in Y,$$

prove that $H_p(f(A)) \le C^p H_p(g(A))$ for every $A \subseteq Y$.

Hausdorff dimension

Problem 15

Let $A \subseteq X$. Prove that

(a) if $H_p(A) < \infty$, then $H_q(A) = 0$ for every $q > p$;

(b) if $H_p(A) > 0$, then $H_q(A) = \infty$ for every $q < p$.

Definition

By Problem 15, for every $A \subseteq X$ the numbers

$$\inf\{p \geq 0 : H_p(A) = 0\} \quad \text{and} \quad \sup\{p \geq 0 : H_p(A) = \infty\}$$

are equal. Their common value is called the **Hausdorff dimension** of A and is denoted by $\dim_H(A)$.

Problem 16

If $X = \mathbb{R}^d$ prove that there is a constant $\gamma_d > 0$ such that $\gamma_d H_d$ is the Lebesgue measure on $\mathbb{R}^d$. In particular, $\dim_H(\mathbb{R}^d) = d$.