Lecture 5

Lebesgue measure on $\mathbb{R}^d$ and its properties

Graduate Real Analysis

Fall Semester

d-dimensional Lebesgue measure

Definition

Let $\mathrm{v}_d : J^d \to [0, \infty]$ be the volume function on $\mathbb{R}^d$. The d-dimensional **Lebesgue outer measure** is defined for any $E \subseteq \mathbb{R}^d$ by setting

$$\lambda_d^*(E) = \inf \left\{ \sum_{n\in\mathbb{N}} \mathrm{v}_d(E_n) : (E_n)_{n\in\mathbb{N}} \subseteq J_{co}^d \quad \text{and} \quad E \subseteq \bigcup_{n\in\mathbb{N}} E_n \right\}.$$

- The d-dimensional **Lebesgue measure** λ_d is the restriction of λ_d^* to the Carathéodory σ-algebra $\mathcal{L}(\mathbb{R}^d) = \mathcal{M}(\lambda_d^*)$ of λ_d^*-measurable sets.
- $\mathcal{L}(\mathbb{R}^d) = \mathcal{M}(\lambda_d^*)$ will be called the **Lebesgue σ-algebra**.
- The members of $\mathcal{L}(\mathbb{R}^d)$ are the **Lebesgue measurable sets**.
- By Carathéodory's theorem $(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d), \lambda_d)$ is a complete measure space. For any rectangle $E \in J^d = J_o^d \cup J_{oc}^d \cup J_{co}^d \cup J_c^d$, we have $\lambda_d(E) = \mathrm{v}_d(E)$ and λ_d is σ-finite on $\mathbb{R}^d$, and $\mathrm{Bor}(\mathbb{R}^d) \subseteq \mathcal{L}(\mathbb{R}^d)$.
- If $d = 1$ we shall abbreviate λ_d^* to λ^* and λ_d to λ.

Regularity of the Lebesgue measure

Theorem

For $E \subseteq \mathbb{R}^d$ the following conditions are equivalent:

(i) $E \in \mathcal{L}(\mathbb{R}^d)$;

(ii) For every $\varepsilon > 0$ there exists an open set $O \supseteq E$ so that $\lambda_d^*(O \setminus E) \leq \varepsilon$;

(iii) There exists a G_δ-set $G \supseteq E$ with $\lambda_d^*(G \setminus E) = 0$;

(iv) For every $\varepsilon > 0$ there exists a closed set $C \subseteq E$ with $\lambda_d^*(E \setminus C) \leq \varepsilon$;

(v) There exists an F_σ set $F \subseteq E$ with $\lambda_d^*(E \setminus F) = 0$.

Moreover, if $E \in \mathcal{L}(\mathbb{R}^d)$, then one has

$$\begin{aligned}\lambda_d(E) &= \inf\{\lambda_d(U) : U \supseteq E \text{ and } U \text{ is open}\}\\ &= \sup\{\lambda_d(F) : F \subseteq E \text{ and } F \text{ is closed}\}\\ &= \sup\{\lambda_d(K) : K \subseteq E \text{ and } K \text{ is compact}\}.\end{aligned}$$

Proof: 1/6

Step 1. We show that $\lambda_d^*(E) = \inf\{\lambda_d(U) : U \supseteq E \text{ and } U \text{ is open}\}$ for every $E \subseteq \mathbb{R}^d$. Let $\alpha(E) = \inf\{\lambda_d(U) : U \supseteq E \text{ and } U \text{ is open}\}$.

- By monotonicity $\lambda_d^*(E) \leq \lambda_d(U)$ for every open $U \supseteq E$, so $\lambda_d^*(E)$ is not larger than the infimum, i.e. $\lambda_d^*(E) \leq \alpha(E)$.
- For the converse we may assume that $\lambda_d^*(E) < \infty$, since otherwise there is nothing to do. Let $\varepsilon > 0$ and choose $(I_n)_{n\in\mathbb{N}} \subseteq J_{co}^d$ such that

$$E \subseteq \bigcup_{n\in\mathbb{N}} I_n \quad \text{and} \quad \sum_{n\in\mathbb{N}} \mathrm{v}_d(I_n) \leq \lambda_d^*(E) + \frac{\varepsilon}{2}.$$

- If $I_n = [a_n, b_n)$ we may enlarge it slightly to an open rectangle $U_n \supseteq I_n$ with $\mathrm{v}_d(U_n) \leq \mathrm{v}_d(I_n) + \varepsilon 2^{-n-1}$.
- Then $U = \bigcup_{n\in\mathbb{N}} U_n$ is open, $E \subseteq U$ and, by subadditivity,

$$\alpha(E) \leq \lambda_d(U) \leq \sum_{n\in\mathbb{N}} \mathrm{v}_d(U_n) \leq \sum_{n\in\mathbb{N}} \mathrm{v}_d(I_n) + \frac{\varepsilon}{2} \leq \lambda_d^*(E) + \varepsilon.$$

Proof: 2/6

Step 2. We show that (i) $\implies$ (ii). Let $E \in \mathcal{L}(\mathbb{R}^d)$ and $\varepsilon > 0$.

- Assume first that $\lambda_d(E) < \infty$. By Step 1 there is an open set $O \supseteq E$ with $\lambda_d(O) \leq \lambda_d(E) + \varepsilon$. Since E is measurable and $\lambda_d(E) < \infty$,
$$\lambda_d^*(O \setminus E) = \lambda_d(O) - \lambda_d(E) \leq \varepsilon.$$
- The assumption $\lambda_d(E) < \infty$ is essential here, since otherwise we cannot subtract $\lambda_d(E)$.
- In the general case put $E_n = E \cap (-n, n)^d$ for $n \in \mathbb{N}$. Then we have $\lambda_d(E_n) < \infty$, so by the previous case there are open sets $O_n \supseteq E_n$ with $\lambda_d^*(O_n \setminus E_n) \leq \varepsilon 2^{-n}$.
- Since $E = \bigcup_{n\in\mathbb{N}} E_n$, the set $O = \bigcup_{n\in\mathbb{N}} O_n$ is open, $E \subseteq O$ and $O \setminus E \subseteq \bigcup_{n\in\mathbb{N}}(O_n \setminus E_n)$, hence
$$\lambda_d^*(O \setminus E) \leq \sum_{n\in\mathbb{N}} \lambda_d^*(O_n \setminus E_n) \leq \sum_{n\in\mathbb{N}} \varepsilon 2^{-n} = \varepsilon.$$

Proof: 3/6

Step 3. We show that (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

- Assume (ii). For every $j \in \mathbb{N}$ choose an open set $O_j \supseteq E$ such that $\lambda_d^*(O_j \setminus E) \leq \frac{1}{j}$ and put

$$G = \bigcap_{j \in \mathbb{N}} O_j.$$

- Then G is a G_δ-set, $E \subseteq G$ and $G \setminus E \subseteq O_j \setminus E$ for every $j \in \mathbb{N}$, so by monotonicity

$$\lambda_d^*(G \setminus E) \leq \inf_{j \in \mathbb{N}} \frac{1}{j} = 0,$$

  which is (iii).

- Assume (iii) and let $G \supseteq E$ be a G_δ-set with $\lambda_d^*(G \setminus E) = 0$.
- Then $G \in \mathrm{Bor}(\mathbb{R}^d) \subseteq \mathcal{L}(\mathbb{R}^d)$, and $G \setminus E \in \mathcal{L}(\mathbb{R}^d)$ because λ_d is a complete measure and $\lambda_d^*(G \setminus E) = 0$. Consequently

$$E = G \setminus (G \setminus E) \in \mathcal{L}(\mathbb{R}^d),$$

  which is (i).

Proof: 4/6

Step 4. We show that (i) $\iff$ (iv) and (iii) $\iff$ (v) by passing to complements.

- If $C \subseteq E$ then $O = C^c$ satisfies $O \supseteq E^c$ and

$$O \setminus E^c = O \cap E = E \setminus C,$$

  and C is closed if and only if O is open.
- Hence E satisfies (iv) if and only if E^c satisfies (ii). Since $E \in \mathcal{L}(\mathbb{R}^d)$ if and only if $E^c \in \mathcal{L}(\mathbb{R}^d)$, the equivalence (i) $\iff$ (iv) follows from (i) $\iff$ (ii) applied to E^c.
- Similarly, $F \subseteq E$ is an F_σ-set if and only if $F^c \supseteq E^c$ is a G_δ-set, and $F^c \setminus E^c = E \setminus F$, so E satisfies (v) if and only if E^c satisfies (iii).

Proof: 5/6

Step 5. We prove the first two equalities. Let $E \in \mathcal{L}(\mathbb{R}^d)$.

- Let $\alpha(E) = \inf\{\lambda_d(U) : U \supseteq E \text{ and } U \text{ is open}\}$. By monotonicity $\lambda_d(E) \leq \alpha(E)$, and the reverse inequality is precisely Step 1.
- Let $\beta(E) = \sup\{\lambda_d(C) : C \subseteq E \text{ and } C \text{ is closed}\}$. By monotonicity we also have $\beta(E) \leq \lambda_d(E)$.
- If $\lambda_d(E) < \infty$, then by (iv) for every $\varepsilon > 0$ there is a closed set $C \subseteq E$ with $\lambda_d(E \setminus C) \leq \varepsilon$, whence

$$\beta(E) \geq \lambda_d(C) = \lambda_d(E) - \lambda_d(E \setminus C) \geq \lambda_d(E) - \varepsilon.$$

- If $\lambda_d(E) = \infty$, then by (iv) there is a closed set $C \subseteq E$ with $\lambda_d(E \setminus C) \leq 1$, and since $\lambda_d(E) \leq \lambda_d(C) + \lambda_d(E \setminus C)$ we obtain $\lambda_d(C) = \infty$, which also gives $\beta(E) \geq \lambda_d(E)$.
- In both cases we have $\beta(E) = \lambda_d(E)$.

Proof: 6/6

Step 6. We prove the third equality.

- Set

$$\beta(E) = \sup\{\lambda_d(C) : C \subseteq E \text{ and } C \text{ is closed}\},$$
$$\gamma(E) = \sup\{\lambda_d(K) : K \subseteq E \text{ and } K \text{ is compact}\}.$$

- Every compact subset of $\mathbb{R}^d$ is closed, so $\gamma(E) \leq \beta(E)$.
- Fix a closed set $C \subseteq E$ and put $K_n = C \cap \overline{B(0,n)}$ for $n \in \mathbb{N}$. Each K_n is closed and bounded, hence compact, and $K_n \subseteq K_{n+1} \subseteq C \subseteq E$.
- By continuity from below $\lim_{n\to\infty} \lambda_d(K_n) = \lambda_d(C)$, so for every $n \in \mathbb{N}$ we have $\lambda_d(K_n) \leq \gamma(E)$ and consequently

$$\lambda_d(C) = \lim_{n\to\infty} \lambda_d(K_n) \leq \gamma(E).$$

- Taking the supremum over all closed $C \subseteq E$ we obtain $\beta(E) \leq \gamma(E)$.
- Hence $\beta(E) = \gamma(E)$ and the proof is finished. $\square$

Approximation by finite unions of open rectangles

Theorem

If $E \in \mathcal{L}(\mathbb{R}^d)$ and $\lambda_d(E) < \infty$, then for every $\varepsilon > 0$ there is a set $A \in \mathcal{L}(\mathbb{R}^d)$ which is a finite union of open rectangles such that

$$\lambda_d(E \triangle A) < \varepsilon,$$

where $E \triangle A = (E \setminus A) \cup (A \setminus E)$ is the symmetric difference of E and A.

- In other words, every Lebesgue measurable set of finite measure is, up to a set of arbitrarily small measure, a finite union of open rectangles.

Proof: Let $\varepsilon > 0$. Since $E \in \mathcal{L}(\mathbb{R}^d)$, by the previous theorem there is an open set $U \supseteq E$ with $\lambda_d(U \setminus E) < \frac{\varepsilon}{2}$. In particular

$$\lambda_d(U) \leq \lambda_d(E) + \frac{\varepsilon}{2} < \infty.$$

Proof

- Since U is open, $U \in \mathcal{L}(\mathbb{R}^d)$, so by the previous theorem there is a compact set $K \subseteq U$ with $\lambda_d(U \setminus K) < \frac{\varepsilon}{2}$.
- Every $x \in K$ has an open rectangle R_x with $x \in R_x \subseteq U$, and by compactness finitely many of them cover K. Let

  $$A = R_{x_1} \cup \ldots \cup R_{x_m},$$

  so that $K \subseteq A \subseteq U$ and, by monotonicity,

  $$\lambda_d(U \setminus A) \leq \lambda_d(U \setminus K) < \frac{\varepsilon}{2}.$$

- Since $E \subseteq U$ and $A \subseteq U$ we have $E \setminus A \subseteq U \setminus A$ and $A \setminus E \subseteq U \setminus E$, hence

  $$\lambda_d(E \triangle A) = \lambda_d(E \setminus A) + \lambda_d(A \setminus E) \leq \lambda_d(U \setminus A) + \lambda_d(U \setminus E) < \varepsilon. \quad \square$$

Translation invariance of the Lebesgue measure

Theorem

(i) For every $E \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$ we have

$$\lambda_d^*(x+E) = \lambda_d^*(E), \quad \text{where} \quad x+E = \{x+y : y \in E\}.$$

(ii) For every $E \in \mathcal{L}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we have $x+E \in \mathcal{L}(\mathbb{R}^d)$ and

$$\lambda_d(x+E) = \lambda_d(E).$$

- The key observation is that $x + I \in J_{co}^d$ and

$$\mathrm{v}_d(x+I) = \mathrm{v}_d(I)$$

for every $I \in J_{co}^d$, since a translation shifts all the vertices of a rectangle by the same vector.

Proof of (i)

Proof of (i).

- For every $I \in J_{co}^d$ and $x \in \mathbb{R}^d$ we have $x + I \in J_{co}^d$ and $\mathrm{v}_d(x+I) = \mathrm{v}_d(I)$.
- Take $(I_n)_{n\in\mathbb{N}} \subseteq J_{co}^d$ so that $E \subseteq \bigcup_{n\in\mathbb{N}} I_n$, then for any $x \in \mathbb{R}^d$:
$$x + E \subseteq x + \bigcup_{n\in\mathbb{N}} I_n = \bigcup_{n\in\mathbb{N}} (x + I_n).$$
- Thus
$$\lambda_d^*(x+E) \leq \sum_{n\in\mathbb{N}} \mathrm{v}_d(x+I_n) = \sum_{n\in\mathbb{N}} \mathrm{v}_d(I_n),$$
and taking the infimum over all such coverings we obtain $\lambda_d^*(x+E) \leq \lambda_d^*(E)$ for any $E \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$.
- Applying this result with $-x$ in place of x and $x + E$ in place of E we obtain
$$\lambda_d^*(E) = \lambda_d^*(-x + (x+E)) \leq \lambda_d^*(x+E).$$
- Consequently we obtain $\lambda_d^*(E) = \lambda_d^*(x+E)$ as desired. $\square$

Proof of (ii)

Proof of (ii).

- If $E \in \mathcal{L}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we show that $x + E \in \mathcal{L}(\mathbb{R}^d)$.
- Let $A \subseteq \mathbb{R}^d$ and observe that, by part (i), we have

$$\begin{aligned}
&\lambda_d^*(A \cap (x+E)) + \lambda_d^*(A \cap (x+E)^c) \\
&= \lambda_d^*\big((A \cap (x+E)) - x\big) + \lambda_d^*\big((A \cap (x+E)^c) - x\big) \\
&= \lambda_d^*((A - x) \cap E) + \lambda_d^*((A - x) \cap E^c) \\
&= \lambda_d^*(A - x) \\
&= \lambda_d^*(A),
\end{aligned}$$

where we used that $(x+E)^c = x + E^c$ and that $E \in \mathcal{L}(\mathbb{R}^d)$.

- Thus $x + E \in \mathcal{L}(\mathbb{R}^d)$ and

$$\lambda_d(x+E) = \lambda_d(E),$$

which completes the proof. □

Positive homogeneity of the Lebesgue measure

Theorem

(i) For every $E \subseteq \mathbb{R}^d$ and $\alpha \in \mathbb{R}$ we have

$$\lambda_d^*(\alpha E) = |\alpha|^d \lambda_d^*(E), \quad \text{where} \quad \alpha E = \{\alpha y : y \in E\}.$$

(ii) If $E \in \mathcal{L}(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}$, then $\alpha E \in \mathcal{L}(\mathbb{R}^d)$ and

$$\lambda_d(\alpha E) = |\alpha|^d \lambda_d(E).$$

- Recall from Lecture 4 that $\lambda_d^* = \lambda_{d,o}^*$, so we may compute λ_d^* using coverings by open rectangles.
- This is convenient here, since for $\alpha \neq 0$ we have $\alpha I \in J_o^d$ and $\mathrm{v}_d(\alpha I) = |\alpha|^d \mathrm{v}_d(I)$ for every $I \in J_o^d$, whereas the family J_{co}^d is not preserved by $\alpha < 0$, because $\alpha[a,b) = (\alpha b, \alpha a]$.

Proof of (i)

Proof of (i).

- If $\alpha = 0$, then $\alpha E \subseteq \{0\}$, thus $\lambda_d^*(\alpha E) = 0 = |\alpha|^d \lambda_d^*(E)$, with the convention $0 \cdot \infty = 0$.
- Let $\alpha \neq 0$ and $E \subseteq \mathbb{R}^d$. Let
$$S = \{(I_n)_{n\in\mathbb{N}} : (I_n)_{n\in\mathbb{N}} \subseteq J_o^d\}.$$
- Define a mapping $M_\alpha : S \to S$ by $M_\alpha(s) = (\alpha I_n)_{n\in\mathbb{N}}$ if $s = (I_n)_{n\in\mathbb{N}}$.
- M_α is $1-1$ and onto and its inverse is given by $M_{1/\alpha}$.
- Let
$$S_E = \Big\{(I_n)_{n\in\mathbb{N}} \in S : E \subseteq \bigcup_{n\in\mathbb{N}} I_n\Big\}.$$
Then $M_\alpha[S_E] = S_{\alpha E}$, since
$$E \subseteq \bigcup_{n\in\mathbb{N}} I_n \quad \Longleftrightarrow \quad \alpha E \subseteq \bigcup_{n\in\mathbb{N}} \alpha I_n.$$

Proof of (i)

- Define a set function $\beta : S \to [0, \infty]$ by $\beta(s) = \sum_{n\in\mathbb{N}} \mathrm{v}_d(I_n)$ for $s = (I_n)_{n\in\mathbb{N}}$. Then

$$\lambda_d^*(E) = \inf_{s\in S_E} \beta(s) \quad \text{and} \quad \lambda_d^*(\alpha E) = \inf_{t\in S_{\alpha E}} \beta(t).$$

- Moreover

$$\beta(M_\alpha(s)) = \sum_{n\in\mathbb{N}} \mathrm{v}_d(\alpha I_n) = |\alpha|^d \sum_{n\in\mathbb{N}} \mathrm{v}_d(I_n) = |\alpha|^d \beta(s).$$

- Thus we obtain $\lambda_d^*(\alpha E) = |\alpha|^d \lambda_d^*(E)$, since $M_\alpha[S_E] = S_{\alpha E}$ and

$$\lambda_d^*(\alpha E) = \inf_{t\in S_{\alpha E}} \beta(t) = \inf_{s\in S_E} \beta(M_\alpha(s)) = \inf_{s\in S_E} |\alpha|^d \beta(s) = |\alpha|^d \lambda_d^*(E)$$

as desired. $\square$

Proof of (ii)

Proof of (ii).

- If $\alpha = 0$, then $\alpha E \subseteq \{0\}$ is a null set, hence $\alpha E \in \mathcal{L}(\mathbb{R}^d)$ by the completeness of λ_d and $\lambda_d(\alpha E) = 0 = |\alpha|^d \lambda_d(E)$.
- Let $E \in \mathcal{L}(\mathbb{R}^d)$, $\alpha \neq 0$ and $A \subseteq \mathbb{R}^d$. We show that $\alpha E \in \mathcal{L}(\mathbb{R}^d)$. Note that

$$\lambda_d^*(\alpha^{-1}A) = \lambda_d^*(\alpha^{-1}A \cap E) + \lambda_d^*(\alpha^{-1}A \cap E^c).$$

- Since $\alpha^{-1}A \cap E = \alpha^{-1}(A \cap \alpha E)$ and $\alpha^{-1}A \cap E^c = \alpha^{-1}(A \cap (\alpha E)^c)$, part (i) gives

$$\frac{1}{|\alpha|^d}\lambda_d^*(A) = \frac{1}{|\alpha|^d}\lambda_d^*(A \cap \alpha E) + \frac{1}{|\alpha|^d}\lambda_d^*(A \cap (\alpha E)^c).$$

- Hence $\alpha E \in \mathcal{L}(\mathbb{R}^d)$, since

$$\lambda_d^*(A) = \lambda_d^*(A \cap \alpha E) + \lambda_d^*(A \cap (\alpha E)^c).$$

- By the previous part we also obtain $\lambda_d(\alpha E) = |\alpha|^d \lambda_d(E)$. $\square$

Completion of Lebesgue measure space on $\mathrm{Bor}(\mathbb{R}^d)$

Theorem

The Lebesgue measure space $(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d), \lambda_d)$ *is the completion of the Borel measure space* $(\mathbb{R}^d, \mathrm{Bor}(\mathbb{R}^d), \lambda_d)$.

Proof. Let $\overline{\mathrm{Bor}(\mathbb{R}^d)}$ be the completion of $\mathrm{Bor}(\mathbb{R}^d)$ with respect to λ_d. i.e.

$$\overline{\mathrm{Bor}(\mathbb{R}^d)} = \{A \cup B \subseteq \mathbb{R}^d : A \in \mathrm{Bor}(\mathbb{R}^d) \text{ and } B \subseteq C \in \mathrm{Bor}_0(\mathbb{R}^d)\},$$

where $\mathrm{Bor}_0(\mathbb{R}^d) = \{C \in \mathrm{Bor}(\mathbb{R}^d) : \lambda_d(C) = 0\}$. Since $(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d), \lambda_d)$ is complete and $\mathrm{Bor}(\mathbb{R}^d) \subseteq \mathcal{L}(\mathbb{R}^d)$ thus

$$\overline{\mathrm{Bor}(\mathbb{R}^d)} \subseteq \mathcal{L}(\mathbb{R}^d).$$

Conversely, if $E \in \mathcal{L}(\mathbb{R}^d)$ then $E = F \cup N$, where F is an F_σ set and $\lambda_d(N) = 0$ thus $E \in \overline{\mathrm{Bor}(\mathbb{R}^d)}$ and consequently

$$\mathcal{L}(\mathbb{R}^d) \subseteq \overline{\mathrm{Bor}(\mathbb{R}^d)}. \qquad \square$$

Lebesgue–Stieltjes measure is a Borel measure

Definition

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right-continuous function. The **Lebesgue–Stieltjes outer measure** is defined for any $E \subseteq \mathbb{R}$ by

$$\mu_F^*(E) = \inf \left\{ \sum_{n \in \mathbb{N}} (F(b_n) - F(a_n)) : E \subseteq \bigcup_{n \in \mathbb{N}} (a_n, b_n] \right\}.$$

- The **Lebesgue–Stieltjes measure** μ_F is the restriction of μ_F^* to the Carathéodory σ-algebra $\mathcal{L}_{\mu_F} = \mathcal{M}(\mu_F^*)$ of μ_F^*-measurable sets.
- $\mathcal{L}_{\mu_F} = \mathcal{M}(\mu_F^*)$ will be called the **Lebesgue–Stieltjes σ-algebra**.
- The members of $\mathcal{L}_{\mu_F}$ are the **Lebesgue–Stieltjes measurable sets**.
- By Carathéodory's theorem $(\mathbb{R}, \mathcal{L}_{\mu_F}, \mu_F)$ is a complete measure space.
- Moreover, $(\mathbb{R}, \mathcal{L}_{\mu_F}, \mu_F)$ is a σ-finite measure space and $\mathrm{Bor}(\mathbb{R}) \subseteq \mathcal{L}_{\mu_F}$ and $\mu_F((a, b]) = F(b) - F(a)$ for any $-\infty \leq a \leq b \leq \infty$.

Regularity of the Lebesgue–Stieltjes measure

Problem 1

For $E \subseteq \mathbb{R}$ prove that $\mu_F^*(E) = \inf\left\{ \sum_{n\in\mathbb{N}} \mu_F((a_n, b_n)) : E \subseteq \bigcup_{n\in\mathbb{N}} (a_n, b_n) \right\}$.

Using this show that the following conditions are equivalent:

- (i) $E \in \mathcal{L}_{\mu_F}$;
- (ii) For every $\varepsilon > 0$ there exists an open set $O \supseteq E$ so that $\mu_F^*(O \setminus E) \leq \varepsilon$;
- (iii) There exists a G_δ-set $G \supseteq E$ with $\mu_F^*(G \setminus E) = 0$;
- (iv) For every $\varepsilon > 0$ there exists a closed set $C \subseteq E$ with $\mu_F^*(E \setminus C) \leq \varepsilon$;
- (v) There exists an F_σ set $F \subseteq E$ with $\mu_F^*(E \setminus F) = 0$.

Moreover, if $E \in \mathcal{L}_{\mu_F}$, then one has

$$\begin{aligned}
\mu_F(E) &= \inf\{\mu_F(U) : U \supseteq E \text{ and } U \text{ is open}\} \\
&= \sup\{\mu_F(F) : F \subseteq E \text{ and } F \text{ is closed}\} \\
&= \sup\{\mu_F(K) : K \subseteq E \text{ and } K \text{ is compact}\}.
\end{aligned}$$

Outer measures induced by general set functions

Outer measures induced by set functions

Let $X \neq \emptyset$ be a set and let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection containing $\emptyset, X$. Let $\rho : \mathcal{E} \to [0, \infty]$ be a set function such that $\rho(\emptyset) = 0$. For any $A \subseteq X$

$$\mu^*(A) = \inf \left\{ \sum_{j \in \mathbb{N}} \rho(E_j) : (E_j)_{j \in \mathbb{N}} \subseteq \mathcal{E} \text{ and } A \subseteq \bigcup_{j \in \mathbb{N}} E_j \right\}$$

defines an outer measure, which we call an **outer measure induced by** ρ.

- By Carathéodory's theorem the collection of μ^*-measurable sets

  $$\mathcal{M}(\mu^*) = \left\{ A \subseteq X : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \ \text{ for all } \ E \subseteq X \right\}$$

  is a σ-algebra, which will be called the **Carathéodory σ-algebra**.
- Moreover, $(X, \mathcal{M}(\mu^*), \mu^*)$ is a complete measure space, and the outer measure μ^* restricted to $\mathcal{M}(\mu^*)$ will be called the **Carathéodory measure**.

The sets F_σ, G_δ, $F_{\sigma\delta}$, $G_{\delta\sigma}, \ldots$

Definition

Let X be a set and let $\mathcal{E} \subseteq \mathcal{P}(X)$. We define

- $\mathcal{E}_\sigma = \{\bigcup_{n\in\mathbb{N}} E_n : (E_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}\}$ - the collection of countable unions of sets from $\mathcal{E}$.
- $\mathcal{E}_\delta = \{\bigcap_{n\in\mathbb{N}} E_n : (E_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}\}$ - the collection of countable intersections of sets from $\mathcal{E}$.
- $\mathcal{E}_{\sigma\delta} = (\mathcal{E}_\sigma)_\delta$ - the collection of countable intersections of sets from $\mathcal{E}_\sigma$.
- $\mathcal{E}_{\delta\sigma} = (\mathcal{E}_\delta)_\sigma$ - the collection of countable unions of sets from $\mathcal{E}_\delta$.
- $\mathcal{E}^c = \{E^c : E \in \mathcal{E}\}$ - the collection of complements of sets from $\mathcal{E}$.

Note that $(\mathcal{E}_\sigma)^c = (\mathcal{E}^c)_\delta$ and $(\mathcal{E}_\delta)^c = (\mathcal{E}^c)_\sigma$.

Outer measures induced by general set functions

Problem 2

Let X, $\mathcal{E}$, ρ and μ^* be as above. Prove that μ^* is an outer measure on X. Where do we use the assumption that $\emptyset, X \in \mathcal{E}$?

Problem 3

Let $\mathcal{E} \subseteq \mathcal{P}(X)$. Prove that

$$(\mathcal{E}_\sigma)^c = (\mathcal{E}^c)_\delta, \quad (\mathcal{E}_\delta)^c = (\mathcal{E}^c)_\sigma \quad \text{and} \quad (\mathcal{E}_{\sigma\delta})^c = (\mathcal{E}^c)_{\delta\sigma}.$$

- The Lebesgue outer measure λ_d^* is the outer measure induced by the volume function v_d on $\mathcal{E} = J_o^d$; in this case
  - $\mathcal{E}_{\sigma\delta}$ plays the role of the G_δ-sets;
  - $(\mathcal{E}^c)_{\delta\sigma}$ the role of the F_σ-sets.

Standing assumptions for Problems 3–7

Assumptions

Let $X \neq \emptyset$ be a set and let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection containing $\emptyset, X$. Let $\rho : \mathcal{E} \to [0, \infty]$ be a set function such that $\rho(\emptyset) = 0$, and for $A \subseteq X$ let

$$\mu^*(A) = \inf \left\{ \sum_{j \in \mathbb{N}} \rho(E_j) : (E_j)_{j \in \mathbb{N}} \subseteq \mathcal{E} \text{ and } A \subseteq \bigcup_{j \in \mathbb{N}} E_j \right\}.$$

Let $\mathcal{M}(\mu^*)$ be the Carathéodory σ-algebra. In the problems below we assume, as indicated in each of them, that

(a) $\mu^* = \rho$ on $\mathcal{E}$;

(b) $\mathcal{E} \subseteq \mathcal{M}(\mu^*)$;

(c) μ^* is a σ-finite measure on $(X, \mathcal{M}(\mu^*))$.

Outer regularity of outer measures

Problem 4

Suppose that (a) holds. Prove that

(i) for any $A \subseteq X$ and $\varepsilon > 0$ there exists a set $U \supseteq A$ such that

$$U \in \mathcal{E}_\sigma \quad \text{and} \quad \mu^*(A) \leq \mu^*(U) \leq \mu^*(A) + \varepsilon;$$

(ii) for any $A \subseteq X$ there exists a set $B \supseteq A$ such that

$$B \in \mathcal{E}_{\sigma\delta} \quad \text{and} \quad \mu^*(A) = \mu^*(B).$$

Problem 5

Suppose that (a) holds. Prove that μ^* is **outer regular** with respect to the family $\mathcal{E}_\sigma$, that is, for every $E \subseteq X$ one has

$$\mu^*(E) = \inf\{\mu^*(U) : U \supseteq E \text{ and } U \in \mathcal{E}_\sigma\}.$$

Regularity of the Carathéodory measures

Problem 6

Suppose that (a), (b) and (c) hold. Prove that for $E \subseteq X$ the following conditions are equivalent:

(i) $E \in \mathcal{M}(\mu^*)$;

(ii) for every $\varepsilon > 0$ there exists a set $U \supseteq E$ such that

$$U \in \mathcal{E}_\sigma \quad \text{and} \quad \mu^*(U \setminus E) \leq \varepsilon;$$

(iii) there exists a set $G \supseteq E$ such that

$$G \in \mathcal{E}_{\sigma\delta} \quad \text{and} \quad \mu^*(G \setminus E) = 0;$$

(iv) for every $\varepsilon > 0$ there exists a set $C \subseteq E$ such that

$$C \in (\mathcal{E}^c)_\delta \quad \text{and} \quad \mu^*(E \setminus C) \leq \varepsilon;$$

(v) there exists a set $F \subseteq E$ such that

$$F \in (\mathcal{E}^c)_{\delta\sigma} \quad \text{and} \quad \mu^*(E \setminus F) = 0.$$

Outer and inner regularity of the Carathéodory measures

Problem 7

Suppose that assumptions (a), (b) and (c) hold. Prove that

(i) μ^* is **outer regular** with respect to the family $\mathcal{E}_\sigma$, that is,

$$\mu^*(E) = \inf\{\mu^*(U) : U \supseteq E \text{ and } U \in \mathcal{E}_\sigma\} \quad \text{for any} \quad E \in \mathcal{M}(\mu^*);$$

(ii) μ^* is **inner regular** with respect to the family $(\mathcal{E}^c)_\delta$, that is,

$$\mu^*(E) = \sup\{\mu^*(C) : C \subseteq E \text{ and } C \in (\mathcal{E}^c)_\delta\} \quad \text{for any} \quad E \in \mathcal{M}(\mu^*).$$

- Compare with the regularity theorem for λ_d proved earlier in this lecture.

Approximations

Problem 8

Suppose that assumptions (a) and (b) hold and that $(X, \mathcal{M}(\mu^*), \mu^*)$ is a finite measure space. Prove that for every $E \in \mathcal{M}(\mu^*)$ and every $\varepsilon > 0$ there exists a set $A \in \mathcal{E}_\sigma$ such that

$$\mu^*(E \triangle A) < \varepsilon.$$

In fact one can take A to be a finite union of elements of $\mathcal{E}$, which is the important part of the statement.

- For $X = \mathbb{R}^d$, $\mathcal{E} = J_o^d$ and $\rho = \mathrm{v}_d$ this is the theorem on approximation of sets of finite measure by finite unions of rectangles.
- We now deduce an analogue for the Lebesgue–Stieltjes measures:

Theorem

If $E \in \mathcal{L}_{\mu_F}$ and $\mu_F(E) < \infty$, then for every $\varepsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu_F(E \triangle A) < \varepsilon$.

Outer regularity: equivalent forms

Problem 9

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\mathcal{E} \subseteq \mathcal{M}$ be a collection containing $\emptyset, X$. Consider the following two conditions:

(a) for every $E \in \mathcal{M}$ and every $\varepsilon > 0$ there exists a set $G \in \mathcal{E}$ such that

$$G \supseteq E \quad \text{and} \quad \mu(G \setminus E) \leq \varepsilon;$$

(b) μ is **outer regular** with respect to the family $\mathcal{E}$, i.e.

$$\mu(E) = \inf\{\mu(U) : U \supseteq E \text{ and } U \in \mathcal{E}\} \text{ for any } E \in \mathcal{M}.$$

Prove that

(i) (a) implies (b);

(ii) if $\mu(X) < \infty$, then (b) implies (a);

(iii) if μ is σ-finite and $\mathcal{E}$ is closed under countable unions, then (b) implies (a).

Inner regularity: equivalent forms

Problem 10

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\mathcal{E} \subseteq \mathcal{M}$ be a collection containing $\emptyset, X$. Consider the following two conditions:

(a) for every $E \in \mathcal{M}$ and every $\varepsilon > 0$ there exists a set $F \in \mathcal{E}$ such that

$$F \subseteq E \quad \text{and} \quad \mu(E \setminus F) \leq \varepsilon;$$

(b) μ is **inner regular** with respect to the family $\mathcal{E}$, i.e.

$$\mu(E) = \sup\{\mu(C) : C \subseteq E \text{ and } C \in \mathcal{E}\} \;\text{ for any }\; E \in \mathcal{M}.$$

Prove that

(i) (a) implies (b);

(ii) if $\mu(X) < \infty$, then (b) implies (a);

(iii) if μ is σ-finite and $\mathcal{E}$ is closed under countable unions, then (b) implies (a).

Regularity and approximation by null sets

Problem 11

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $\mathcal{E} \subseteq \mathcal{M}$ be a collection containing $\emptyset, X$. Prove that

(i) if μ is outer regular with respect to $\mathcal{E}$ and $\mathcal{E}$ is closed under countable unions, then for every $E \in \mathcal{M}$ there exists a set $G \in \mathcal{E}_\delta$ such that
$$G \supseteq E \quad \text{and} \quad \mu(G \setminus E) = 0;$$

(ii) if μ is inner regular with respect to $\mathcal{E}$, then for every $E \in \mathcal{M}$ there exists a set $F \in \mathcal{E}_\sigma$ such that
$$F \subseteq E \quad \text{and} \quad \mu(E \setminus F) = 0.$$

- Note the asymmetry: in (ii) no closure assumption on $\mathcal{E}$ is needed.