Lecture 6

Cantor set and Cantor function
Nonmeasurable sets and Kakeya sets

Graduate Real Analysis

Fall Semester

Some properties of Lebesgue measure

Let λ be the Lebesgue measure on $\mathbb{R}$. Then we have

1. $\lambda(\{x\}) = 0$ for any $x \in \mathbb{R}$,
2. $\mathbb{Q}$ has Lebesgue measure 0. In fact, $\lambda(U) = 0$ for any countable set $U \subseteq \mathbb{R}$,
3. If U is open in $\mathbb{R}$, then $\lambda(U) > 0$,
4. If E is a null set in $\mathbb{R}$ (i.e. $\lambda(E) = 0$) then E^c is dense in $\mathbb{R}$.

Proof.

For any open interval $I \subseteq \mathbb{R}$ we have $\lambda(I) > 0$. Since $\lambda(E) = 0$ this means that E cannot contain any open interval. Thus $E^c \cap I \neq \emptyset$ for any interval $I \subseteq \mathbb{R}$ but this means that E^c is dense in $\mathbb{R}$. □

Nowhere dense set with positive Lebesgue measure

5. Let $\mathbb{Q} \cap [0,1] = \{r_j : j \in \mathbb{N}\}$ and given $\varepsilon > 0$ let I_j be the open interval of length $\varepsilon 2^{-j}$, which is centered at r_j. Then the set

$$U = (0,1) \cap \bigcup_{j=1}^{\infty} I_j$$

is open and dense in $\mathbb{R}$, since $U \supseteq \mathbb{Q} \cap (0,1)$. Thus U is topologically large but small in the sense of measure, since

$$\lambda(U) \leq \sum_{j=1}^{\infty} \lambda(I_j) \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

6. If $K = [0,1] \setminus U$ then K is closed and nowhere dense thus topologically small but large in the sense of measure since

$$\lambda(K) = 1 - \lambda(U) \geq 1 - \varepsilon.$$

Accumulation point, isolated point, perfect set

Accumulation point

Let (X, ρ) be a metric space, $x \in X$ is called **an accumulation point of** $E \subseteq X$ if for every open set $U \ni x$ we have

$$(E \setminus \{x\}) \cap U \neq \emptyset.$$

An accumulation point x of $E \subseteq X$ is sometimes also called **a limit point of** E or **a cluster point of** E.

Isolated point

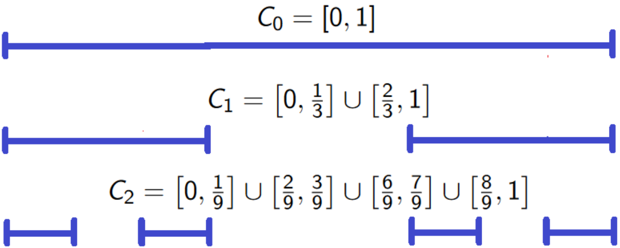
A point $x \in E$ is called **an isolated point of** E if it is not an accumulation point of E.

Perfect sets

We say that a subset E of a metric space (X, ρ) is **perfect** if E is closed and every point of E is its limit point.

There exists a perfect set in $\mathbb{R}$ which contains no segment.

- Let $C_0 = [0, 1]$. Given C_n that consists of 2^n disjoint closed intervals each of length 3^{-n} take each of these intervals and delete the open middle third to produce two closed intervals each of length 3^{-n-1}.



- Take C_{n+1} to be the union of 2^{n+1} closed intervals so formed and continue.

Cantor set

Cantor set

The set

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$$

is called **the Cantor set** or **ternary Cantor set**.

- Each $C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$ is closed and bounded thus compact, and the family $(C_n)_{n\in\mathbb{N}}$ has finite intersection property thus the Cantor set is compact and $\mathcal{C} \neq \emptyset$.

Property (*)

By the construction for each $k, m \in \mathbb{N}$ we see that

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \cap \mathcal{C} = \emptyset.$$

Properties of the Cantor set

- Every segment (α, β) contains a segment of the form (*) if m is sufficiently large, since the set

$$\left\{ \frac{\ell}{3^m} : m \in \mathbb{N} \text{ and } 0 \leq \ell \leq 3^m - 1 \right\}$$

is dense in $[0,1]$. Thus $\mathcal{C}$ contains no segment (α, β). This also shows that $\operatorname{int} \mathcal{C} = \emptyset$.

- To prove that $\mathcal{C}$ is perfect it is enough to show that $\mathcal{C}$ contains no isolated points. Let $x \in \mathcal{C}$ and let I_n be the unique interval from C_n which contains $x \in I_n$. Let x_n be the endpoint of I_n such that $x \neq x_n$. It follows from the construction of $\mathcal{C}$ that $x_n \in \mathcal{C}$. Hence x is a limit point of $\mathcal{C}$ thus $\mathcal{C}$ is perfect.

Cantor set has Lebesgue measure zero

- We have

$$\lambda(\mathcal{C}) = 0.$$

Indeed, by the construction note that

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$$

and

$$\lambda(C_n) = \left(\frac{2}{3}\right)^n \quad \text{for any} \quad n \in \mathbb{N}.$$

By the continuity of λ, since $C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$ and $\lambda(C_0) = 1$, we have

$$\lambda(\mathcal{C}) = \lim_{n\to\infty} \lambda(C_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

More about Cantor set

- Each component of C_n can be described as the set

$$C_n = \left\{ \sum_{n=1}^{\infty} \frac{\varepsilon_j}{3^j} \ : \ \varepsilon_j \in \{0,1,2\} \text{ and } \varepsilon_j \neq 1 \text{ for } 1 \leq j \leq n \right\}.$$

- Consequently,

$$\mathcal{C} = \left\{ \sum_{n=1}^{\infty} \frac{\varepsilon_j}{3^j} \ : \ \varepsilon_j \in \{0,2\} \right\}.$$

Fact

Fact

Any number $\sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j}$ is uniquely determined by its sequence $\varepsilon = (\varepsilon_j)_{j\in\mathbb{N}}$ with $\varepsilon_j \in \{0, 2\}$.

Proof. Take $\varepsilon = (\varepsilon_j)_{j\in\mathbb{N}}$, $\delta = (\delta_j)_{j\in\mathbb{N}}$ with $\varepsilon_j, \delta_j \in \{0, 2\}$ such that $\varepsilon \neq \delta$. Let $N = \min\{j \in \mathbb{N} \ : \ \varepsilon_j \neq \delta_j\}$ and assume $0 = \varepsilon_N < \delta_N = 2$. Then

$$\sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j} = \sum_{j=1}^{N-1} \frac{\varepsilon_j}{3^j} + \sum_{j=N+1}^{\infty} \frac{\varepsilon_j}{3^j} \leq \sum_{j=1}^{N-1} \frac{\delta_j}{3^j} + \frac{2}{3^{N+1}} \sum_{j=0}^{\infty} \frac{1}{3^j}$$

$$\leq \sum_{j=1}^{N-1} \frac{\delta_j}{3^j} + \frac{2}{3^{N+1}} \underbrace{\frac{1}{1-\frac{1}{3}}}_{\frac{3}{2}} = \sum_{j=1}^{N-1} \frac{\delta_j}{3^j} + \frac{1}{3^N} < \sum_{j=1}^{N-1} \frac{\delta_j}{3^j} + \frac{2}{3^N} \leq \sum_{j=1}^{\infty} \frac{\delta_j}{3^j}.$$

This completes the proof. □

Remarks

Remark

We have two different representations

$$\frac{1}{3} = \sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j} = A, \quad \varepsilon_1 = 1, \quad \varepsilon_j = 0 \quad \text{for} \quad j \geq 2.$$

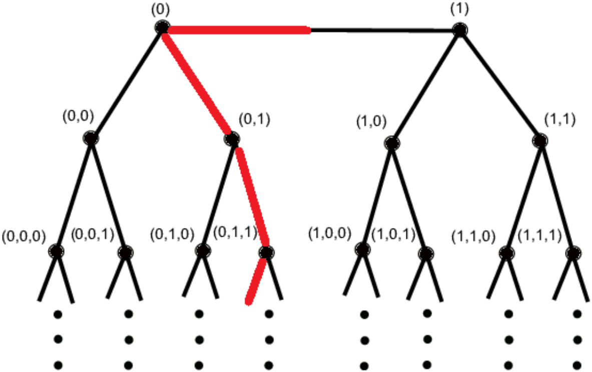
$$\frac{1}{3} = \sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j} = B, \quad \varepsilon_1 = 0, \quad \varepsilon_j = 2 \quad \text{for} \quad j \geq 2.$$

There is a bijection $\phi : \{0,1\}^{\mathbb{N}} \to \mathcal{C}$ defined by

$$\phi(z) = \frac{2}{3} \sum_{j=0}^{\infty} \frac{z_j}{3^j} \quad \text{for} \quad z = (z_j)_{j \in \mathbb{N}}, \quad z_j \in \{0,1\},$$

and consequently $\mathrm{card}\,(\mathcal{C}) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}}) = \mathrm{card}\,(\mathbb{R}) = \mathfrak{c}.$

Cantor tree



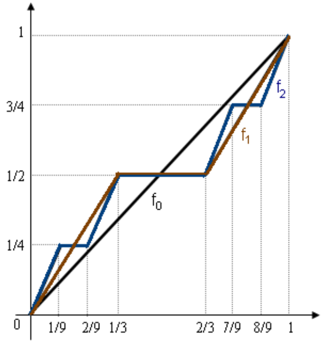
$$\varepsilon = (0, 1, 1, 0, \varepsilon_4, \varepsilon_5, \ldots)$$

Cantor–Lebesgue function

Cantor–Lebesgue approximating function

For each $n \in \mathbb{N}$ and $x \in [0,1]$ we define

$$f_n(x) = \left(\frac{3}{2}\right)^n \lambda(C_n \cap [0,x]).$$



Cantor–Lebesgue function

- Because C_n is a finite union of intervals,

$$f_n(x) = \left(\frac{3}{2}\right)^n \lambda(C_n \cap [0,x]).$$

  is a polygonal function with $f_n(0) = 0$, $f_n(1) = 1$, and f_n is constant on each of $2^n - 1$ open intervals composing $[0,1] \setminus C_n$ and rises with slope $\left(\frac{3}{2}\right)^n$ on each on the 2^n closed intervals composing C_n.
- If the j-th interval of C_n counting from the left is $[a_{n,j}, b_{n,j}]$, then

$$f_n(a_{n,j}) = \frac{j-1}{2^n} \quad \text{and} \quad f_n(b_{n,j}) = \frac{j}{2^n}.$$

  Also $a_{n,j} = a_{n+1,2j-1}$ and $b_{n,j} = b_{n+1,2j}$ and

$$f_{n+1}(a_{n,j}) = f_n(a_{n,j}) \quad \text{and} \quad f_{n+1}(b_{n,j}) = f_n(b_{n,j})$$

  and f_{n+1} agrees with f_n at all of the endpoints of the intervals of C_n and therefore on $[0,1] \setminus C_n$.

Cantor–Lebesgue function

- Within any particular interval $[a_{n,j}, b_{n,j}]$ of C_n the greatest difference between $f_n(x)$ and $f_{n+1}(x)$ is at the new endpoints within that interval and the magnitude of the difference is at most $\frac{1}{6 \cdot 2^n}$. Thus we have

$$|f_{n+1}(x) - f_n(x)| \leq \frac{1}{6 \cdot 2^n} \quad \text{for any} \quad n \in \mathbb{N}.$$

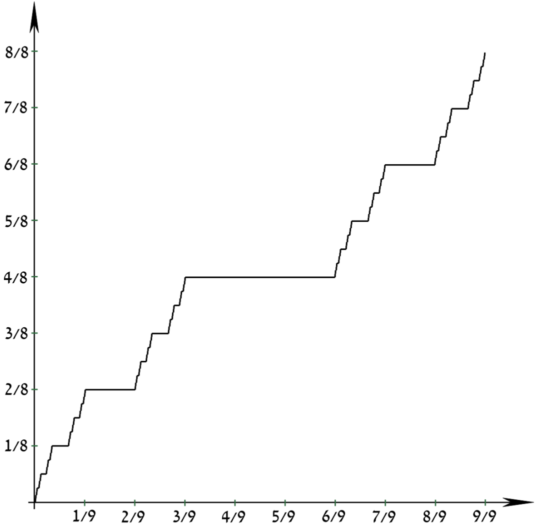
- Since

$$\frac{1}{6} \sum_{n \in \mathbb{N}} \frac{1}{2^n} < \infty.$$

  thus $(f_n)_{n \in \mathbb{N}}$ is uniformly convergent to a continuous function $f : [0, 1] \to [0, 1]$.
- This function is called the **Cantor–Lebesgue function** or **Devil's staircase function**.

Cantor–Lebesgue function

The **Cantor–Lebesgue function** or **Devil's staircase function**:



Properties of the Cantor–Lebesgue function

- Every f_n is non-decreasing so is f.
- Let $\phi : \{0,1\}^{\mathbb{N}} \to \mathcal{C}$, be given by

$$\phi(z) = \frac{2}{3} \sum_{j=0}^{\infty} \frac{z_j}{3^j}$$

for every $z = (z_j)_{j \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}$. Then

$$f(\phi(z)) = \frac{1}{2} \sum_{j=0}^{\infty} \frac{z_j}{2^j}.$$

- Indeed, fix $\eta = \{\eta_0, \eta_1, \ldots\} \in \{0,1\}^{\mathbb{N}}$ and for each $n \in \mathbb{N}$ take I_n to be the component of C_n containing $\phi(\eta)$. Then I_{n+1} will be the left-hand third of I_{n+1} if $\eta_n = 0$ and the right-hand third if $\eta_n = 1$.

Properties of the Cantor–Lebesgue function

- Taking a_n to be the left-hand endpoint of I_n we see that

$$a_{n+1} = a_n + \frac{2}{3} \cdot \frac{\eta_n}{3^n} \quad \text{and} \quad f_{n+1}(a_{n+1}) = f_n(a_n) + \frac{1}{2} \cdot \frac{\eta_n}{2^n}$$

for each $n \in \mathbb{N}$. Thus

$$\phi(\eta) = \lim_{n\to\infty} a_n$$

and

$$f(\phi(z)) = \lim_{n\to\infty} f_n(a_n) = \frac{1}{2} \sum_{j=0}^{\infty} \frac{\eta_j}{2^j}.$$

- In particular, $f[\mathcal{C}] = [0,1]$ since any $x \in [0,1]$ is expressible as

$$x = \sum_{j=0}^{\infty} \frac{\eta_j}{2^{j+1}} = f(\phi(\eta))$$

for some $z = (z_j)_{j\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$.

The axiom of choice

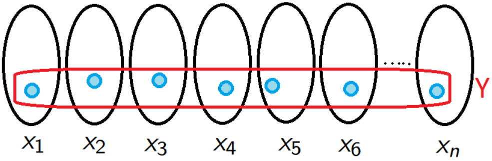
The axiom of choice

If $(X_\alpha)_{\alpha \in A}$ is a nonempty collection of nonempty sets, then

$$\prod_{\alpha \in A} X_\alpha \neq \emptyset.$$

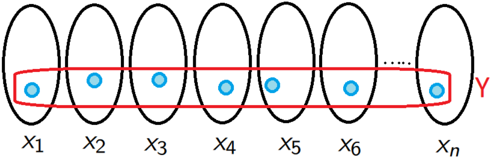
Corollary

If $(X_\alpha)_{\alpha \in A}$ is a disjoint collection of nonempty sets, then there is a set $Y \subseteq \bigcup_{\alpha \in A} X_\alpha$ (called **the selector** of $(X_\alpha)_{\alpha \in A}$) such that $Y \cap X_\alpha$ contains precisely one element for each $\alpha \in A$.



Proof

Proof.



Take $f \in \prod_{\alpha \in A} X_\alpha \neq \emptyset$. Define

$$Y = f[A],$$

then

$$Y \cap X_\alpha = \{f(\alpha)\}$$

since $f(\alpha) \in X_\alpha$. $\square$

Some equivalence relation

Equivalence relation

We write $x \sim y$ iff $x - y \in \mathbb{Q}$. This is an equivalence relation:

1. $x \sim x$, since $x - x = 0 \in \mathbb{Q}$ for any $x \in [0,1]$,
2. if $x \sim y$, then $y \sim x$,
3. if $x \sim y$ and $y \sim z$, then $x \sim z$.

Let $\{\epsilon_\alpha : \alpha \in A\}$ be the set of all equivalence classes. The equivalence classes ϵ_α are either disjoint or coincide, and $[0,1]$ is the disjoint union of all equivalence classes, we write

$$[0,1] = \bigcup_{\alpha \in A} \epsilon_\alpha,$$

and

$$\epsilon_\alpha \cap \epsilon_\beta = \emptyset \quad \text{if} \quad \alpha \neq \beta.$$

Nonmeasurable subset of $\mathbb{R}$

Vitali's Theorem

There exists a nonmeasurable subset of $\mathbb{R}$.

Proof. Since $[0,1] = \bigcup_{\alpha \in A} \epsilon_\alpha$, by the axiom of choice there is a selector

$$\mathcal{N} = \{x_\alpha : \alpha \in A\}$$

for the family $(\epsilon_\alpha)_{\alpha \in A}$ such that

$$\text{card}\,(\mathcal{N} \cap \epsilon_\alpha) = 1 \quad \text{for any} \quad \alpha \in A$$

We claim that the set $\mathcal{N}$ is **non-measurable**.

- Let $\{r_k : k \in \mathbb{N}\} = \mathbb{Q} \cap [0,1]$. For each $k \in \mathbb{N}$ consider

$$\mathcal{N}_k = r_k + \mathcal{N}.$$

Proof 1/2

- We show that
  1. $\mathcal{N}_k \cap \mathcal{N}_l = \emptyset$ if $k \neq l$,
  2. $[0,1] \subseteq \bigcup_{k=1}^{\infty} \mathcal{N}_k \subseteq [-1,2]$.
- **Proof of (i).** If $\mathcal{N}_k \cap \mathcal{N}_l \neq \emptyset$ for some $l \neq k$, then there are $\alpha, \beta \in A$ and $r_k \neq r_l$ such that

$$x_\alpha + r_k = x_\beta + r_l \iff x_\alpha - x_\beta = r_l - r_k.$$

  Consequently, $\alpha \neq \beta$ and $x_\alpha - x_\beta \in \mathbb{Q}$. Hence, $x_\alpha \sim x_\beta$. But this contradicts the fact that $\mathcal{N}$ contains only **one** representative of each equivalence class.
- **Proof of (ii).** Of course $\bigcup_{k=1}^{\infty} \mathcal{N}_k \subseteq [-1,2]$. If $x \in [0,1]$ then $x \sim x_\alpha$ for some $\alpha \in A$ and $x - x_\alpha = r_k$ for some $k \in \mathbb{N}$ thus

$$x = x_\alpha + r_k \in r_k + \mathcal{N} = \mathcal{N}_k.$$

Proof 2/2

- If we assume that $\mathcal{N}$ is measurable then

$$\lambda(\mathcal{N}) = \lambda(r_k + \mathcal{N}) = \lambda(\mathcal{N}_k) \quad \text{for all} \quad k \in \mathbb{N},$$

by the translation invariance of Lebesgue measure. Now note that

$$\begin{aligned} 1 = \lambda([0,1]) &\leq \lambda\left(\bigcup_{k=1}^{\infty} \mathcal{N}_k\right) \\ &= \sum_{k=1}^{\infty} \lambda(\mathcal{N}_k) \\ &\leq \lambda([-1,2]) = 3. \end{aligned}$$

- This is a contradiction since neither $\lambda(\mathcal{N}) = 0$ nor $\lambda(\mathcal{N}) = 1$. $\square$

Kakeya sets

Definition

A set $\mathcal{K} \subseteq \mathbb{R}^d$ is called a **Kakeya set** (or a **Besicovitch set**) if it contains a unit line segment in every direction, that is, for every $e \in \mathbb{R}^d$ with length one, i.e. $|e| = 1$, there exists $y \in \mathbb{R}^d$ such that

$$\{y + te : t \in [0, 1]\} \subseteq \mathcal{K}.$$

- In 1917 Kakeya asked for the smallest area of a region in the plane inside which a unit needle can be rotated by 180°.
- Besicovitch proved in 1919 that there exist Kakeya sets in $\mathbb{R}^2$ of Lebesgue measure zero, so no smallest area exists.
- Such sets are nevertheless large in the sense of Lecture 4: the Kakeya conjecture asserts that every Kakeya set in $\mathbb{R}^d$ has Hausdorff dimension d. This is classical for $d = 2$ and was recently proved for $d = 3$ by Wang and Zahl.

Bishop's construction of a Kakeya set in $\mathbb{R}^2$

Theorem

There exists a compact Kakeya set in $\mathbb{R}^2$ of Lebesgue measure zero.

Proof:

- The construction below produces a compact set $\mathcal{K} \subseteq \mathbb{R}^2$ with $\lambda_2(\mathcal{K}) = 0$ which contains a unit segment of every slope in $[0, 1]$.
- The union of four copies of $\mathcal{K}$ rotated by $0, \frac{\pi}{4}, \frac{\pi}{2}$ and $\frac{3\pi}{4}$ is then a compact Kakeya set of measure zero, since every direction of $\mathbb{R}^2$ is obtained from a slope in $[0, 1]$ by one of these four rotations.

Step 1. The sequence $(x_k)_{k=0}^{\infty}$. Let

$$x_k = \begin{cases} 0 & \text{if } k = 0, \\ \frac{k}{2^n} - 1 & \text{if } k \in [2^n, 2^{n+1}) \text{ and } n \text{ is even}, \\ 2 - \frac{k}{2^n} & \text{if } k \in [2^n, 2^{n+1}) \text{ and } n \text{ is odd}. \end{cases}$$

Proof: 1/5

- Thus

$$(x_k)_{k=0}^\infty = \left(0, 0, 1, \frac{1}{2}, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \frac{7}{8}, \frac{6}{8}, \frac{5}{8}, \frac{4}{8}, \dots\right)$$

  that is, the sequence traverses the dyadic rationals $(j2^{-n})_{j=0}^{2^n}$, reversing the order each time. In particular $(x_k)_{k=0}^\infty$ is dense in $[0,1]$.

- If $k \in [2^n, 2^{n+1})$, then

$$\varepsilon(k) := |x_{k+1} - x_k| = \frac{1}{2^n},$$

  so $\varepsilon(k) \geq \varepsilon(k+1)$ for $k \geq 1$ and $\lim_{k\to\infty} \varepsilon(k) = 0$.

- Moreover, the intervals $[x_k - \varepsilon(k), x_k + \varepsilon(k)]$ cover $[0,1]$ infinitely often: for every $k \in \mathbb{N}$ and every $b \in [0,1]$ there is $N > k$ with $b \in [x_N - \varepsilon(N), x_N + \varepsilon(N)]$.

Proof: 2/5

Step 2. The function f and the set $\mathcal{K}$. Let $\{s\} = s - \lfloor s \rfloor$ denotes the fractional part of $s \in \mathbb{R}$. For $t \in [0,1]$ let $f(t) = \lim_{k\to\infty} f_k(t)$, where

$$f_k(t) = \sum_{j=1}^{k} \{2^j t\} \frac{x_{j-1} - x_j}{2^j}.$$

Define the set $\mathcal{K} = \{(\alpha, f(t) + \alpha t) : \alpha, t \in [0,1]\}$.

- The series converges uniformly, since $|\{2^j t\}(x_{j-1} - x_j)| \le 1$ for every $j \in \mathbb{N}$, so f is a well defined bounded function and consequently $\mathcal{K}$ is a bounded subset of $\mathbb{R}^2$.
- $\mathcal{K}$ contains a unit segment of every slope $t \in [0,1]$: for a fixed $t \in [0,1]$ the points $(\alpha, f(t) + \alpha t)$ with $\alpha \in [0,1]$ form the segment joining $(0, f(t))$ and $(1, f(t) + t)$, whose slope is t and whose length is $\sqrt{1+t^2} \ge 1$.
- It remains to show that $\lambda_2(\mathrm{cl}(\mathcal{K})) = 0$; the set $\mathrm{cl}(\mathcal{K})$ is then compact, and it still contains a unit segment of every slope in $[0,1]$.

Proof: 3/5

Step 3. On each dyadic interval the function $f_k(t) + x_k t$ is constant.

- Fix $k \in \mathbb{N}$ and let I be a component of $[0,1] \setminus 2^{-k}\mathbb{Z}$, that is, $I = \left(\frac{m}{2^k}, \frac{m+1}{2^k}\right)$ for some $m \in \{0, 1, \ldots, 2^k - 1\}$.
- If $t \in I$ and $1 \leq j \leq k$, then $\{2^j t\} = 2^j t - \lfloor 2^j t \rfloor$ is affine on I with slope 2^j, hence f_k is affine on I and, by telescoping,

$$f_k'(t) = \sum_{j=1}^{k} (x_{j-1} - x_j) = x_0 - x_k = -x_k, \quad \text{since} \quad x_0 = 0.$$

- Consequently

$$f_k(t) + x_k t \quad \text{is constant on each component of} \quad [0,1] \setminus 2^{-k}\mathbb{Z}.$$

- Note also that

$$|f(t) - f_k(t)| \leq \sum_{i>k} \{2^i t\} \frac{|x_{i-1} - x_i|}{2^i} \leq \varepsilon(k) \sum_{i>k} \frac{1}{2^i} = \frac{\varepsilon(k)}{2^k},$$

because $|x_{i-1} - x_i| = \varepsilon(i-1) \leq \varepsilon(k)$ for every $i > k$.

Proof: 4/5

Step 4. The diameter estimate. Fix $n \in \mathbb{N}$ and $k \in [2^n, 2^{n+1})$, so that $\varepsilon(k) = 2^{-n}$, and let $\alpha \in [0,1]$ satisfy $|\alpha - x_k| \leq \varepsilon(k)$.

- For t in a component I of $[0,1] \setminus 2^{-k}\mathbb{Z}$ we write

$$f(t) + \alpha t = \big(f(t) - f_k(t)\big) + \big(f_k(t) + x_k t\big) + t(\alpha - x_k).$$

- The first term varies by at most $2\varepsilon(k)2^{-k}$ by Step 3, the second one is constant on I by Step 3, and the third one varies by at most $|I|\,|\alpha - x_k| \leq 2^{-k}\varepsilon(k)$.
- Hence the image of I under $t \mapsto f(t) + \alpha t$ has diameter at most

$$2\varepsilon(k)2^{-k} + 2^{-k}\varepsilon(k) = \frac{3}{2^{n+k}}.$$

- If moreover α varies in an interval $J \subseteq [x_k - \varepsilon(k), x_k + \varepsilon(k)]$ with $|J| = 2^{-n-k}$, then the third term changes by at most $|t|\,|J| \leq 2^{-n-k}$, so

$$\{f(t) + \alpha t : t \in I,\ \alpha \in J\} \quad \text{is contained in a box of side length} \quad \frac{4}{2^{n+k}}.$$

Proof: 5/5

Step 5. The covering argument. Let $\mathcal{K}_J = \{(\alpha, f(t) + \alpha t) : \alpha \in J,\ t \in [0,1]\}$ with n, k and J be as in Step 4.

- There are 2^k components I of $[0,1] \setminus 2^{-k}\mathbb{Z}$ and $2^k + 1$ remaining dyadic points, and each of them contributes to $\mathcal{K}_J$ a set contained in a closed square of side $4 \cdot 2^{-n-k}$, by Step 4. Hence

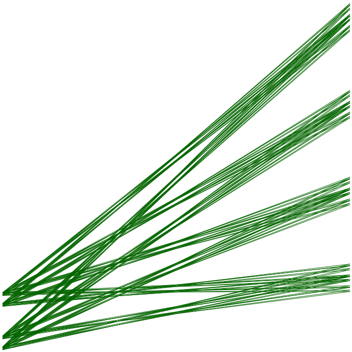
$$\lambda_2^*(\mathcal{K}_J) \leq (2^{k+1} + 1)\frac{16}{2^{2n+2k}} = O\Big(\frac{1}{2^{2n+k}}\Big).$$

- The interval $[x_k - \varepsilon(k), x_k + \varepsilon(k)]$ is the union of 2^{k+1} such intervals J, and as k runs over $[2^n, 2^{n+1})$ the points x_k run over all the dyadic rationals $j2^{-n}$, so these 2^n intervals cover $[0,1]$. Therefore

$$\lambda_2^*(\mathcal{K}) \leq 2^n \cdot 2^{k+1} \cdot O\Big(\frac{1}{2^{2n+k}}\Big) = O\Big(\frac{1}{2^n}\Big).$$

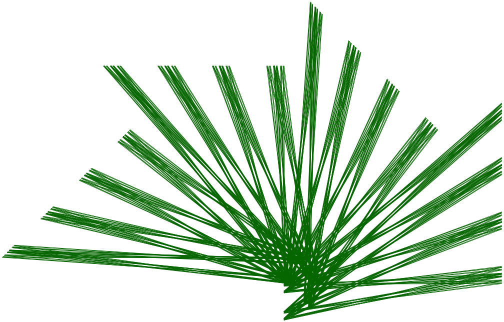
- For every n the set $\mathcal{K}$ is covered by finitely many closed squares of total area $O(2^{-n})$, so the same is true for $\mathrm{cl}(\mathcal{K})$, and $\lambda_2(\mathrm{cl}(\mathcal{K})) = 0$ by letting $n \to \infty$.
- Finally, $\mathrm{cl}(\mathcal{K})$ is compact and the union of its four copies rotated by $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ is a compact Kakeya set of Lebesgue measure zero. $\square$

The set $\mathcal{K}$



- The segments joining $(0, f(t))$ and $(1, f(t) + t)$ for 160 values of $t \in [0, 1]$; the whole set $\mathcal{K}$ is the union of all of them, and it has Lebesgue measure zero.

Four rotated copies of $\mathcal{K}$



- Rotating $\mathcal{K}$ by $0, \frac{\pi}{4}, \frac{\pi}{2}$ and $\frac{3\pi}{4}$ we obtain unit segments in every direction, while the union still has Lebesgue measure zero.

Generalized Cantor sets

Construction

Let I be a bounded interval and let $\alpha \in (0,1)$. The **open middle α-th** of I is the open interval with the same midpoint as I and with length $\alpha|I|$. Let $(\alpha_j)_{j\in\mathbb{N}} \subseteq (0,1)$. We define a decreasing sequence $(K_j)_{j\geq 0}$ of closed sets as follows:

- $K_0 = [0,1]$;
- K_j is obtained by removing the open middle α_j-th from each of the intervals that make up K_{j-1}.

The set $K = \bigcap_{j\in\mathbb{N}} K_j$ is called a **generalized Cantor set**.

- For $\alpha_j = \frac{1}{3}$ we recover the ternary Cantor set from this lecture.

Generalized Cantor sets

Problem 1

Let K be the generalized Cantor set determined by a sequence $(\alpha_j)_{j\in\mathbb{N}} \subseteq (0,1)$. Show that

a) K is compact, perfect and nowhere dense;

b) $\mathrm{card}(K) = \mathfrak{c}$;

c) if $\alpha_j = \frac{1}{3}$ for all $j \in \mathbb{N}$, then K is the ternary Cantor set and $\lambda(K) = 0$;

d) in general

$$\lambda(K) = \prod_{j=1}^{\infty}(1-\alpha_j),$$

so $\lambda(K) > 0$ if and only if $\sum_{j=1}^{\infty} \alpha_j < \infty$. Show also that for every $\beta \in (0,1)$ one can choose $(\alpha_j)_{j\in\mathbb{N}}$ so that $\lambda(K) = \beta$.

Lebesgue measurable sets which are not Borel

Problem 2

Assuming that $\mathrm{card}(\mathrm{Bor}(\mathbb{R})) = \mathfrak{c}$ and using Problem 1, show that not every Lebesgue measurable subset of $\mathbb{R}$ is a Borel set.

Problem 3

Consider $\mathbb{R}$ with the Lebesgue measure λ. Construct a set $X \subseteq \mathbb{R}$ such that for every nonempty open set $V \subseteq \mathbb{R}$ we have

$$\lambda(V \cap X) > 0 \quad \text{and} \quad \lambda(V \cap X^c) > 0.$$

Hint: use Problem 1.

Density of measurable sets and Steinhaus' theorem

Problem 4

Let $E \subseteq \mathbb{R}$ be Lebesgue measurable with $\lambda(E) > 0$. Show that for every $\alpha \in (0, 1)$ there exists an open interval I such that

$$\lambda(E \cap I) > \alpha\lambda(I).$$

Problem 5 (Steinhaus' theorem)

Let $E \subseteq \mathbb{R}$ be Lebesgue measurable with $\lambda(E) > 0$. Show that the set

$$E - E = \{x - y : x, y \in E\}$$

contains an interval centered at 0. Hint: if I is as in Problem 4 with $\alpha > \frac{3}{4}$, then $E - E$ contains $\left(-\frac{1}{2}\lambda(I), \frac{1}{2}\lambda(I)\right)$.

Perfect subsets of sets of positive measure

Problem 6 (Cantor–Bendixson theorem)

Show that every closed set $A \subseteq \mathbb{R}$ is the union of a perfect set P and a countable set C. The set P is allowed to be empty.

Problem 7

Let $E \subseteq \mathbb{R}$ be Lebesgue measurable with $0 < \lambda(E) \leq \infty$. Show that for every $q \in (0, \lambda(E))$ there exists a perfect set $B \subseteq E$ such that

$$\lambda(B) = q.$$

Hint: use the inner regularity of λ and Problem 6.

Two consequences

Problem 8

Show that every Lebesgue measurable set $A \subseteq \mathbb{R}$ with $\lambda(A) > 0$ has the cardinality of the continuum.
Hint: use Problem 7.

Problem 9

Let $A \subseteq [a, b]$ be Lebesgue measurable with $\lambda(A) > 0$. Show that there exist $x, y \in A$ such that $|x - y|$ is an irrational number.
Hint: use Problem 8.

- Problem 9 also follows immediately from Steinhaus' theorem.