Lecture 7

Measurable functions

Graduate Real Analysis

Fall Semester

Induced mapping

Any mapping $f : X \to Y$ between two sets X and Y induces a mapping $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by

$$f^{-1}[E] = \{x \in X : f(x) \in E\} \subseteq X \quad \text{for every} \quad E \subseteq Y.$$

This mapping preserves unions, intersections, and complements, i.e.

$$f\Big[\bigcup_{\alpha\in A} E_\alpha\Big] = \bigcup_{\alpha\in A} f[E_\alpha],$$
$$f\Big[\bigcap_{\alpha\in A} E_\alpha\Big] = \bigcap_{\alpha\in A} f[E_\alpha],$$
$$f[E^c] = f[E]^c.$$

If $\mathcal{N}$ is a σ-algebra on Y, then $\{f^{-1}[E] : E \in \mathcal{N}\}$ is a σ-algebra on X.

Measurable mapping

Definition (Measurable mapping)

If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces, a mapping $f : X \to Y$ is called $(\mathcal{M}, \mathcal{N})$-**measurable** or just **measurable** when $\mathcal{M}$ and $\mathcal{N}$ are understood, if

$$f^{-1}[E] \in \mathcal{M} \quad \text{for all} \quad E \in \mathcal{N}.$$

Remark

The composition of measurable mappings is measurable. More precisely, if $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable and $g : Y \to Z$ is $(\mathcal{N}, \mathcal{O})$-measurable then

$$g \circ f : X \to Z$$

is $(\mathcal{M}, \mathcal{O})$-measurable.

Useful facts

Proposition

If $\mathcal{N}$ is generated by $\mathcal{E}$, then $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable iff

$$f^{-1}[E] \in \mathcal{M} \quad \textit{for all} \quad E \in \mathcal{E}.$$

Proof ($\Longrightarrow$). If $f : X \to Y$ is measurable then obviously $f^{-1}[E] \in \mathcal{M}$ for all $E \in \mathcal{E}$ since $\mathcal{N} = \sigma(\mathcal{E}) \supseteq \mathcal{E}$.

Proof ($\Longleftarrow$). Observe that $\mathcal{O} = \{E \subseteq Y : f^{-1}[E] \in \mathcal{M}\}$ is a σ-algebra that contains $\mathcal{E}$ thus $\mathcal{N} = \sigma(\mathcal{E}) \subseteq \mathcal{O}$. □

Corollary

If X and Y are metric or topological spaces, every continuous function $f : X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable. Here we abbreviate $\mathrm{Bor}(Z)$ to $\mathcal{B}_Z$.

Proof. f is continuous iff $f^{-1}[U]$ is open in X for every open $U \subseteq Y$. □

Measurable functions

Definition (Measurable function)

If $(X, \mathcal{M})$ is measurable space, a real-valued or complex-valued function f on X will be called $\mathcal{M}$-**measurable** or just **measurable** if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ or $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ measurable.

In particular, $f : \mathbb{R} \to \mathbb{C}$ is **Lebesgue** (resp. **Borel**) **measurable** if it is $(\mathcal{L}(\mathbb{R}), \mathcal{M}_{\mathbb{C}})$ (resp. $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{C}})$) measurable, likewise for $f : \mathbb{R} \to \mathbb{R}$.

Remark (Warning!)

If $f, g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable it does not follow that $f \circ g$ is Lebesgue measurable, even if g is assumed continuous.

- If $E \in \mathcal{B}_{\mathbb{R}}$ we have $f^{-1}[E] \in \mathcal{L}(\mathbb{R})$, but unless $f^{-1}[E] \in \mathcal{B}_{\mathbb{R}}$ there is no guarantee that $g^{-1}\left[f^{-1}[E]\right]$ will be in $\mathcal{L}(\mathbb{R})$.
- However, is f is Borel measurable then $f \circ g$ is Lebesgue or Borel whenever g is.

Proposition

Proposition

If $(X, \mathcal{M})$ is a measurable space and $f : X \to \mathbb{R}$ the following are equivalent:

- (a) f *is* $\mathcal{M}$*-measurable,*
- (b) $f^{-1}[(a, \infty)] \in \mathcal{M}$ *for all* $a \in \mathbb{R}$*,*
- (c) $f^{-1}[[a, \infty)] \in \mathcal{M}$ *for all* $a \in \mathbb{R}$*,*
- (d) $f^{-1}[(-\infty, a)] \in \mathcal{M}$ *for all* $a \in \mathbb{R}$*,*
- (e) $f^{-1}[(-\infty, a]] \in \mathcal{M}$ *for all* $a \in \mathbb{R}$*.*

Proof. It follows from the previous proposition and the fact that

$$\begin{aligned} \mathcal{B}_{\mathbb{R}} &= \sigma(\{(a, \infty) : a \in \mathbb{R}\}) = \sigma(\{[a, \infty) : a \in \mathbb{R}\}) \\ &= \sigma(\{(-\infty, a) : a \in \mathbb{R}\}) = \sigma(\{(-\infty, a] : a \in \mathbb{R}\}). \end{aligned}$$ □

Measurable functions in products

Definition (Measurable functions in products)

Given a set X, if $\{(Y_\alpha, \mathcal{N}_\alpha) : \alpha \in A\}$ is a family of measurable spaces and $f_\alpha : X \to Y_\alpha$ is a map for each $\alpha \in A$, there is a unique smallest σ-algebra on X with respect to which the f_α's are all measurable, namely the σ-algebra generated by the family

$$\{f_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{N}_\alpha \text{ and } \alpha \in A\}.$$

It is called the σ-algebra generated by the family $\{f_\alpha : \alpha \in A\}$.

Example

If $X = \prod_{\alpha \in A} Y_\alpha$, then

$$\bigotimes_{\alpha \in A} \mathcal{N}_\alpha = \sigma\big(\{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{N}_\alpha \text{ and } \alpha \in A\}\big)$$

is the σ-algebra generated by the coordinate maps $\pi_\alpha : X \to Y_\alpha$.

Useful proposition

Proposition

Let $(X, \mathcal{M})$ and $\{(Y_\alpha, \mathcal{N}_\alpha) : \alpha \in A\}$ be measurable spaces, $Y = \prod_{\alpha \in A} Y_\alpha$,

$$\mathcal{N} = \bigotimes_{\alpha \in A} \mathcal{N}_\alpha = \sigma\Big(\{\pi_\alpha^{-1}[E_\alpha] : E_\alpha \in \mathcal{N}_\alpha \text{ and } \alpha \in A\}\Big),$$

where $\pi_\alpha : Y \to Y_\alpha$ is the coordinate map for $\alpha \in A$. Then $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable iff $f_\alpha = \pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable for all $\alpha \in A$. In particular, $f : X \to \prod_{j=1}^{\infty} X_j$ given by $f(x) = (f_1(x), f_2(x), f_3(x), \ldots)$ is measurable iff all f_j's are measurable.

Proof ($\Longrightarrow$). If $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable and $\pi_\alpha : Y \to Y_\alpha$ is $(\mathcal{N}, \mathcal{N}_\alpha)$-measurable then $f_\alpha = \pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable.

Proof ($\Longleftarrow$). If each f_α is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable then for all $E_\alpha \in \mathcal{N}_\alpha$ we have $f^{-1}[\pi_\alpha^{-1}[E_\alpha]] = f_\alpha^{-1}[E_\alpha] \in \mathcal{M}$, so f is measurable by the previous proposition. $\square$

Corollary

Corollary

Let $(X, \mathcal{M})$ *be a measurable space. A function* $f : X \to \mathbb{C}$ *is* $\mathcal{M}$*-measurable iff* $\mathrm{Re}(f)$ *and* $\mathrm{Im}(f)$ *are* $\mathcal{M}$*-measurable.*

Proof. This follows from the fact that

$$\mathcal{B}_{\mathbb{C}} = \mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}.$$

Moreover, we have

$$f(x) = \mathrm{Re}(f)(x) + i\,\mathrm{Im}(f)(x) \sim (\mathrm{Re}(f)(x), \mathrm{Im}(f)(x)).$$

This completes the proof. $\square$

Extended real numbers system

Definition

If $(X, \mathcal{M})$ is a measurable space, f is a function on X and $E \in \mathcal{M}$ we say f is measurable on E if $f^{-1}[B] \cap E \in \mathcal{M}$ for all Borel sets $B \in \mathcal{B}_X$.

Remark

Let $\overline{\mathbb{R}} = [-\infty, \infty]$ be the extended real number system. We define Borel sets in $\overline{\mathbb{R}}$ by

$$\mathcal{B}_{\overline{\mathbb{R}}} = \{E \subseteq \overline{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}.$$

It can be easily verified that

$$\mathcal{B}_{\overline{\mathbb{R}}} = \sigma\left(\{(a, \infty] : a \in \mathbb{R}\}\right) = \sigma\left(\{[-\infty, a) : a \in \mathbb{R}\}\right),$$

and we define $f : X \to \overline{\mathbb{R}}$ to be $\mathcal{M}$-measurable if it is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

Mesurability is preserved under algebraic operations

Proposition

Let $(X, \mathcal{M})$ be a measurable space. If $f, g : X \to \mathbb{C}$ are $\mathcal{M}$-measurable, then so are $f + g$ and $f \cdot g$.

Proof. Define $F : X \to \mathbb{C} \times \mathbb{C}$, $\phi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\psi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$F(x) = (f(x), g(x)), \qquad \phi(z, w) = z + w, \qquad \psi(z, w) = z \cdot w.$$

Since

$$\mathcal{B}_{\mathbb{C}\times\mathbb{C}} = \mathcal{B}_{\mathbb{C}} \otimes \mathcal{B}_{\mathbb{C}},$$

F is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}\times\mathbb{C}})$-measurable by the previous proposition. The functions ϕ and ψ are $(\mathcal{B}_{\mathbb{C}\times\mathbb{C}}, \mathcal{B}_{\mathbb{C}})$-measurable, since ϕ, ψ are continuous. Thus

$$f + g = \phi \circ F, \quad \text{and} \quad f \cdot g = \psi \circ F$$

are $\mathcal{M}$-measurable. □

Mesurability is preserved under limiting operations

Proposition

Let $(X, \mathcal{M})$ *be a measurable space. If* $(f_n)_{n\in\mathbb{N}}$ *is a sequence of* $\overline{\mathbb{R}}$*-valued measurable functions, then the functions*

$$g_1(x) = \sup_{n\in\mathbb{N}} f_n(x), \quad g_3(x) = \limsup_{n\to\infty} f_n(x),$$
$$g_2(x) = \inf_{n\in\mathbb{N}} f_n(x), \quad g_4(x) = \liminf_{n\to\infty} f_n(x),$$

are all measurable. If

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in X$*, then* f *is measurable.*

Proof 1/2

We have to show $g_1^{-1}((a,\infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

$$\begin{aligned} g_1^{-1}\big[(a,\infty]\big] &= \{x \in X : g_1(x) \in (a,\infty]\} \\ &= \{x \in X : \sup_{n\in\mathbb{N}} f_n(x) > a\} \\ &= \bigcup_{n\in\mathbb{N}} \{x \in X : f_n(x) > a\} \\ &= \bigcup_{n\in\mathbb{N}} f_n^{-1}\big[(a,\infty]\big] \in \mathcal{M}. \end{aligned}$$

Similarly

$$g_2^{-1}\big[[-\infty,a)\big] = \bigcup_{n\in\mathbb{N}} f_n^{-1}\big[[-\infty,a)\big] \in \mathcal{M}.$$

Thus g_1, g_2 are measurable.

Proof 2/2

Observe that

$$g_3(x) = \limsup_{n\to\infty} f_n(x) = \lim_{k\to\infty} \sup_{n>k} f_n(x) = \lim_{k\to\infty} h_k(x),$$

where $h_k(x) = \sup_{n>k} f_n(x)$, and we have

$$h_{k+1}(x) \leq h_k(x), \qquad \text{thus} \qquad \lim_{k\to\infty} h_k(x) = \inf_{k\in\mathbb{N}} h_k(x).$$

From the previous slide $h_k(x) = \sup_{n>k} f_n(x)$ is $\mathcal{M}$-measurable and $\inf_{k\in\mathbb{N}} h_k(x)$ is also $\mathcal{M}$-measurable. Thus

$$g_3(x) = \limsup_{n\to\infty} f_n(x)$$

is $\mathcal{M}$-measurable, and likewise for g_4. Finally, if $f(x) = \lim_{n\to\infty} f_n(x)$ exists then $f(x) = g_3(x) = g_4(x)$, so f is $\mathcal{M}$-measurable. □

Two corollaries

Corollary

If $f, g : X \to \overline{\mathbb{R}}$ *are measurable, then so are* $\max(f, g)$ *and* $\min(f, g)$.

Corollary

If $(f_n)_{n\in\mathbb{N}}$ *is a sequence of complex-valued measurable functions and*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for all $x \in X$, *then* f *is measurable.*

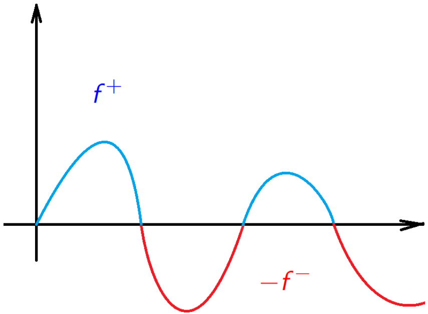
Positive f^+ and negative f^- parts of f

Definition

If $f : X \to \overline{\mathbb{R}}$ we define the **positive** f^+ and **negative** f^- **parts** of f to be

$$f^+(x) = \max(f(x), 0), \quad \text{and} \quad f^-(x) = \max(-f(x), 0).$$

Then $f = f^+ - f^-$. If f is measurable, so are f^+ and f^-.



Polar decomposition

Definition

If $f : X \to \mathbb{C}$, we have its **polar decomposition**

$$f = \operatorname{sgn}(f)|f|,$$

where

$$\operatorname{sgn}(z) = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark

If f is measurable, so are $|f|$ and $\operatorname{sgn}(f)$.

- If f is measurable, so are $|f|$ and $\operatorname{sgn}(f)$. Indeed, $z \mapsto |z|$ is continuous on $\mathbb{C}$, and $z \mapsto \operatorname{sgn}(x)$ is continuous except at the origin.
- If $U \subseteq \mathbb{C}$ is open $\operatorname{sgn}^{-1}[U]$ is either open or of the form $V \cup \{0\}$, where V is open, so sgn is Borel measurable.

Characteristic functions and simple functions

Definition (Characteristic function)

Suppose that $(X, \mathcal{M})$ is a measurable space. If $E \subseteq X$, the **characteristic function** of E (sometimes called the indicator function of E) is defined by

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

$\mathbb{1}_E$ is measurable iff $E \in \mathcal{M}$.

Definition (Simple functions)

A **simple function** on X is a finite linear combination, with complex coefficients, of characteristic functions of sets in $\mathcal{M}$. We do not allow simple functions to assume the values $\pm\infty$. Equivalently, $f : X \to \mathbb{C}$ is simple iff f is measurable and the range of f is a finite subset of $\mathbb{C}$.

Standard representation of simple functions

Standard representation of simple functions

Assume that the range of f is $\{z_1, \dots, z_n\} \subset \mathbb{C}$. Then we can write

$$f = \sum_{j=1}^{n} z_j \mathbb{1}_{E_j}, \quad \text{where} \quad E_j = f^{-1}[\{z_j\}].$$

This is called **the standard representation** of f. It exhibits f a linear combination, with distinct coefficients, of characteristic functions of disjoint sets whose union is X, i.e.

$$E_j \cap E_i = \emptyset \quad \text{if} \quad i \neq j, \quad \bigcup_{j=1}^{n} E_j = X.$$

Remark

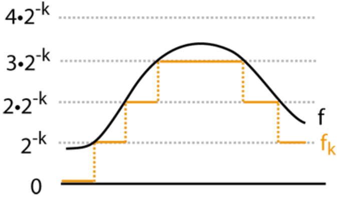
If f and g are simple functions, then so are $f + g$ and $f \cdot g$.

Measurable functions are limits of simple functions

Theorem

Let $(X, \mathcal{M})$ be a measurable space.

- (a) If $f : X \to [0, \infty]$ is measurable then there is a sequence $(\phi_n)_{n\in\mathbb{N}}$ of simple functions such that $0 \leq f_1 \leq f_2 \leq \ldots \leq f$ and $f_n \xrightarrow[n\to\infty]{} f$ pointwise, and $f_n \xrightarrow[n\to\infty]{} f$ uniformly on any set on which f is bounded.
- (b) If $f : X \to \mathbb{C}$ is measurable then there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions such that $0 \leq |f_1| \leq |f_2| \leq \ldots \leq |f|$ and $f_n \xrightarrow[n\to\infty]{} f$ pointwise, and $f_n \xrightarrow[n\to\infty]{} f$ uniformly on any set on which f is bounded.



Proof

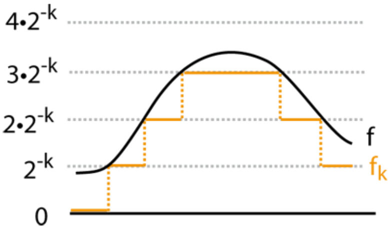
Proof of (a). For $n \in \{0, 1, 2, \ldots\}$ and $0 \le k \le 2^{2n} - 1$, let

$$E_n^k = f^{-1}\big[(k2^{-n}, (k+1)2^{-n}]\big] \quad \text{and} \quad F_n = f^{-1}\big[(2^n, \infty]\big],$$

$$f_n(x) = \sum_{k=0}^{2^{2n}-1} k2^{-n} \mathbb{1}_{E_n^k}(x) + 2^n \mathbb{1}_{F_n}(x).$$

For $n \in \mathbb{N}$, one can easily see

$$f_n(x) \le f_{n+1}(x) \le f(x).$$



Proof

Moreover, on the set F_n^c (since $f(x) \leq 2^n$), we have

$$0 \leq f(x) - f_n(x) = \sum_{k=0}^{2^{2n}-1} (f(x) - k2^{-n})\mathbb{1}_{E_n^k}(x) \leq \sum_{k=0}^{2^{2n}-1} 2^{-n}\mathbb{1}_{E_n^k}(x) \leq 2^{-n}.$$

Proof of (b). If $f = g + ih$ we apply part (a) to the positive and negative parts of g and h obtaining sequences

$$g_n^+ \xrightarrow[n\to\infty]{} g^+, \qquad g_n^- \xrightarrow[n\to\infty]{} g^-,$$

$$h_n^+ \xrightarrow[n\to\infty]{} h^+, \qquad h_n^- \xrightarrow[n\to\infty]{} h^-,$$

of nonnegative increasing simple functions. Taking

$$f_n = g_n^+ - g_n^- + i(h_n^+ - h_n^-),$$

we can easily verify that $(f_n)_{n\in\mathbb{N}}$ has the desired properties. $\square$

Proposition

Proposition

Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Then the following are true.

(a) *If f is measurable and $f = g$ μ-a.e., then g is measurable.*

(b) *If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions and $f_n \xrightarrow[n\to\infty]{} f$ μ-a.e., then f is measurable.*

Remark

- If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions, then the set $D_e = \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable, and $\lim_{n\to\infty} f_n(x)$ is measurable on D_e.
- If μ is not complete and $f_n \xrightarrow[n\to\infty]{} f$ μ-a.e., then one can perturb f on the set

$$D_{ne} = \{x \in X : \lim_{n\to\infty} f_n(x) \text{ does not exist}\},$$

in a way that f is not measurable on X.

Proposition

Proposition

Let $(X, \mathcal{M}, \mu)$ be a measurable space and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If f is $\overline{\mathcal{M}}$-measurable function on X, there is $\mathcal{M}$-measurable function g such that $f = g$ $\overline{\mu}$-a.e.

Proof. From the definition of $\overline{\mu}$ it is obvious if $f = \mathbb{1}_E$, where $E \in \mathcal{M}$, and hence if f is an $\overline{\mathcal{M}}$-measurable simple function.

- For the general case, choose a sequence $(f_n)_{n\in\mathbb{N}}$ of $\overline{\mathcal{M}}$-measurable simple functions that converge pointwise to f by the previous theorem.
- For each $n \in \mathbb{N}$ let g_n be an $\mathcal{M}$-measurable simple function with $f_n = g_n$ except on a set $E_n \in \overline{\mathcal{M}}$ with $\overline{\mu}(E_n) = 0$.
- Choose $N \in \mathcal{M}$ such that $\mu(N) = 0$ and $N \supseteq \bigcup_{n\in\mathbb{N}} E_n$ and set $g = \lim_{n\to\infty} \mathbb{1}_{N^c} g_n$. Then g is $\mathcal{M}$-measurable and $g = f$ on N^c. $\square$

Completeness and measurability

Problem 1

Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that each of the following implications is valid if and only if the measure μ is complete:

a) if f is measurable and $f = g$ μ-a.e., then g is measurable;

b) if f_n is measurable for every $n \in \mathbb{N}$ and $f_n \xrightarrow[n\to\infty]{} f$ μ-a.e., then f is measurable.

- One implication is the proposition proved above. For the converse take a set $N \in \mathcal{M}$ with $\mu(N) = 0$ and $A \subseteq N$, and consider the functions $f = 0$ and $g = \mathbb{1}_A$.

Continuous images of null sets

Problem 2

Show that there exist a strictly increasing continuous function f on $[0,1]$ and a compact set $K \subseteq [0,1]$ such that

$$\lambda_1(K) = 0 \quad \text{and} \quad \lambda_1(f[K]) = 1.$$

- Hint: let c be the Cantor–Lebesgue function from Lecture 6, let K be the Cantor set and consider $f(x) = x + c(x)$.
- In particular, continuous functions need not map null sets to null sets, and therefore they need not map Lebesgue measurable sets to Lebesgue measurable sets.

Continuous images and preimages of measurable sets

Problem 3

Show that there exist a set $E \in \mathcal{L}(\mathbb{R})$ and a continuous function f on an interval containing E such that

$$f[E] \notin \mathcal{L}(\mathbb{R}).$$

Hint: with f and K as in Problem 2, the set $f[K]$ has positive measure, so it contains a nonmeasurable set A; consider $E = f^{-1}[A] \cap K$.

Problem 4

Show that there exists a real-valued $\mathcal{L}(\mathbb{R})$-measurable function g for which

$$g^{-1}[E] \notin \mathcal{L}(\mathbb{R}) \quad \text{for some} \quad E \in \mathcal{L}(\mathbb{R}).$$

Hint: take $g = f^{-1}$, where f is the function from Problem 2, and E from Problem 3.

Lebesgue measurable sets which are not Borel

Problem 5

Show that there exists a set $E \in \mathcal{L}(\mathbb{R})$ such that

$$E \notin \mathrm{Bor}(\mathbb{R}).$$

Hint: the set E from Problem 3 works. If E were a Borel set, then $f[E]$ would be a Borel set as well, since f is a homeomorphism of $[0, 1]$ onto $f[[0, 1]]$.

- Compare with Lecture 6, where the same conclusion was obtained by a cardinality argument. The present proof is constructive and, in addition, exhibits a null set which is not Borel.