Lecture 8

Lebesgue integral

Graduate Real Analysis

Fall Semester

Integration of simple functions

We fix a measurable space $(X, \mathcal{M}, \mu)$ and define

$$L^0_+(X) = \{f : X \to [0, \infty] : f \text{ is measurable}\}.$$

Definition

If $\phi \in L^0_+(X)$ is a simple function of the form $\phi = \sum_{j=1}^n a_j \mathbb{1}_{E_j}$ we define **the integral of ϕ with respect to μ** by

$$\int_X \phi(x) d\mu(x) = \sum_{j=1}^n a_j \mu(E_j) \in [0, \infty],$$

with the convention that $0 \cdot \infty = 0$.

We shall also write

$$\int_X \phi(x) d\mu(x) = \int \phi(x) d\mu(x) = \int \phi d\mu = \int \phi.$$

Integration of simple functions

If $\phi = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j}$ and $A \in \mathcal{M}$ then $\phi \cdot \mathbb{1}_A$ is also simple function,

$$\phi \cdot \mathbb{1}_A = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j \cap A}.$$

Define

$$\int_A \phi(x) d\mu(x) = \int_X \phi(x) \mathbb{1}_A(x) d\mu(x).$$

Proposition

Let ϕ, ψ be simple functions in $L^0_+(X)$.

- (a) *If $c \geq 0$, then $\int_X c\phi d\mu = c \int_X \phi d\mu$.*
- (b) $\int_X (\phi + \psi) d\mu = \int_X \phi d\mu + \int_X \psi d\mu$.
- (c) *If $\phi \leq \psi$, then $\int_X \phi d\mu \leq \int_X \psi d\mu$.*
- (d) *The map $A \mapsto \int_A \phi d\mu$ is measurable on $\mathcal{M}$.*

Proof of (a) and (b)

Let

$$\phi = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j}, \quad \text{and} \quad \psi = \sum_{k=1}^{m} b_k \mathbb{1}_{F_k}.$$

Proof of (a). Note that $\int_X c\phi d\mu = \sum_{j=1}^{n} ca_j\mu(E_j) = c\int_X \phi d\mu$. $\square$

Proof of (b). We consider

$$E_j = \bigcup_{k=1}^{m} E_j \cap F_k, \qquad F_k = \bigcup_{j=1}^{n} E_j \cap F_k.$$

since $X = \bigcup_{j=1}^{n} E_j = \bigcup_{k=1}^{m} F_k$ (disjoint unions). Then

$$\int_X \phi d\mu + \int_X \psi d\mu = \sum_{j=1}^{n} a_j\mu(E_j) + \sum_{k=1}^{m} b_k\mu(F_k)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{m}(a_j + b_k)\mu(E_j \cap F_k) = \int_X (\phi + \psi) d\mu. \qquad \square$$

Proof of (c) and (d)

Proof of (c). If $\phi \leq \psi$ then $a_j \leq b_k$ on $E_j \cap F_k \neq \emptyset$. Then

$$\int_X \phi d\mu = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \mu(E_j \cap F_k) \leq \sum_{j=1}^{n} \sum_{k=1}^{m} b_k \mu(E_j \cap F_k) = \int_X \psi d\mu. \qquad \square$$

Proof of (d). For $A \in \mathcal{M}$ define

$$\nu(A) = \int_A \phi d\mu = \int_X \phi \mathbb{1}_A d\mu.$$

Note that ν is countably additive. Indeed, if $(A_k)_{k\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{M}$ and $A = \bigcup_{k=1}^{\infty} A_k$, then

$$\nu(A) = \sum_{j=1}^{n} a_j \mu(A \cap E_j) = \sum_{k\in\mathbb{N}} \sum_{j=1}^{n} a_j \mu(A_k \cap E_j) = \sum_{k\in\mathbb{N}} \int_{A_k} \phi d\mu = \sum_{k\in\mathbb{N}} \nu(A_k),$$

since μ is countably additive. $\square$

Integration of nonnegative measurable functions

Definition

The integral can be extended to all functions $f \in L^0_+(X)$ by defining

$$\int_X f(x)d\mu(x) = \sup\left\{ \int_X \phi(x)d\mu(x) : 0 \leq \phi \leq f, \text{ and } \phi \text{ is simple}\right\}.$$

Remark

- By the previous proposition (see item (c)), the two definitions of $\int f$ agree when f is simple, as the family of simple functions over which the supremum is taken includes f itself.
- If $f \leq g$, then $\int_X f d\mu \leq \int_X g d\mu$.
- For any $c \geq 0$ we have $\int_X cf d\mu = c \int_X f d\mu$.

Monotone convergence theorem

Theorem (Monotone convergence theorem (MCT))

Let $(X, \mathcal{M}, \mu)$ *be a measure space. If* $(f_n)_{n\in\mathbb{N}}$ *is a sequence in* $L^0_+(X)$ *such that* $f_n \leq f_{n+1}$ *for all* $n \in \mathbb{N}$ *and*

$$f = \lim_{n\to\infty} f_n = \sup_{n\in\mathbb{N}} f_n,$$

then

$$\int_X f(x)d\mu(x) = \lim_{n\to\infty} \int_X f_n(x)d\mu(x).$$

In other words,

$$\int \lim_{n\to\infty} f_n(x)d\mu(x) = \lim_{n\to\infty} \int_X f_n(x)d\mu(x).$$

Proof

Proof. Since $(\int_X f_n d\mu)_{n\in\mathbb{N}}$ is an increasing sequence of numbers so the limit exists (possibly equal to ∞). Since $f_n \le f$, then $\int_X f_n d\mu \le \int_X f d\mu$ for all $n \in \mathbb{N}$, and consequently

$$\lim_{n\to\infty} \int_X f_n d\mu \le \int_X f d\mu.$$

We now show that

$$\lim_{n\to\infty} \int_X f_n d\mu \ge \int_X f d\mu.$$

Fix $\alpha \in (0,1)$, and let ϕ be a simple function with $0 \le \phi \le f$. Consider

$$E_n = \{x \in X : f_n(x) \ge \alpha\phi(x)\}.$$

Then $E_n \subseteq E_{n+1}$ for any $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} E_n$, hence

$$\int_X f_n d\mu \ge \int_{E_n} f_n d\mu \ge \alpha \int_{E_n} \phi d\mu.$$

Proof

By the previous proposition (see item (d)) and the continuity of measure

$$\nu(A) = \int_A \phi d\mu$$

we obtain

$$\lim_{n\to\infty} \int_X f_n d\mu \geq \alpha \lim_{n\to\infty} \int_{E_n} \phi d\mu = \alpha \lim_{n\to\infty} \nu(E_n) = \alpha \int_X \phi d\mu.$$

Since $\alpha \in (0,1)$ was arbitrary

$$\lim_{n\to\infty} \int_X f_n d\mu \geq \int_X \phi d\mu.$$

Since ϕ was arbitrary simple function such that $0 \leq \phi \leq f$ thus

$$\lim_{n\to\infty} \int_X f_n d\mu \geq \sup_{0\leq\phi\leq f} \int_X \phi d\mu = \int_X f d\mu. \qquad \square$$

Theorem

Theorem

Let $(X, \mathcal{M}, \mu)$ *be a measure space. If* $(f_n)_{n\in\mathbb{N}}$ *is a finite or infinite sequence in* $L^0_+(X)$ *such that*

$$f = \sum_{n\in\mathbb{N}} f_n,$$

then

$$\int_X f d\mu = \int_X \sum_{n\in\mathbb{N}} f_n d\mu = \sum_{n\in\mathbb{N}} \int_X f_n d\mu.$$

Proof. First we show that

$$\int_X (f_1 + f_2) d\mu = \int_X f_1 d\mu + \int_X f_2 d\mu.$$

We find sequences $(\phi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ of simple functions in $L^0_+(X)$ that increase to f_1 and f_2, then $(\phi_n + \psi_n)_{n\in\mathbb{N}} \in L^0_+(X)$ and increases to $f_1 + f_2$.

Proof

By the **(MCT)** we conclude

$$\int_X (f_1 + f_2)\,d\mu \underbrace{=}_{\textbf{(MCT)}} \lim_{n\to\infty} \int_X (\phi_n + \psi_n)\,d\mu$$

$$= \lim_{n\to\infty} \int_X \phi_n\,d\mu + \lim_{n\to\infty} \int_X \psi_n\,d\mu \underbrace{=}_{\textbf{(MCT)}} \int_X f_1\,d\mu + \int_X f_2\,d\mu.$$

Now by induction

$$\int_X \sum_{n=1}^{N} f_n\,d\mu = \sum_{n=1}^{N} \int_X f_n\,d\mu \quad \text{for any} \quad N \in \mathbb{N}.$$

Letting $N \to \infty$ and applying **(MCT)** again we have

$$\int_X \sum_{n=1}^{\infty} f_n\,d\mu = \int_X \lim_{N\to\infty} \sum_{n=1}^{N} f_n\,d\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n\,d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu. \qquad \square$$

Useful fact

Proposition

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in L^0_+(X)$, then

$$\int_X f d\mu = 0 \quad \Longleftrightarrow \quad f = 0 \text{ a.e.}.$$

Proof. This is clear if f is a simple function. If $f = \sum_{j=1}^n a_j \mathbb{1}_{E_j}$ with $a_j \geq 0$ then

$$0 = \int_X f d\mu = \sum_{j=1}^n a_j \mu(E_j),$$

iff $a_j = 0$ or $\mu(E_j) = 0$ for each $1 \leq j \leq n$. In general, if $f \in L^0_+(X)$ and $f = 0$ a.e. and ϕ is a simple function with $0 \leq \phi \leq f$, then $\phi = 0$ a.e., hence

$$\int_X f d\mu = \sup_{0 \leq \phi \leq f} \int_X \phi d\mu = 0.$$

Proof

On the other hand assume that $\int_X f d\mu = 0$. Note that

$$\{x \in X : f(x) > 0\} = \bigcup_{n \in \mathbb{N}} E_n, \quad \text{where} \quad E_n = \{x \in X : f(x) > n^{-1}\}.$$

If it is false that $f = 0$ a.e., then we must have

$$0 < \mu(\{x \in X : f(x) > 0\}) = \lim_{n \to \infty} \mu(E_n),$$

hence there must be at least one $n \in \mathbb{N}$ so that $\mu(E_n) > 0$. But then

$$\int_X f d\mu \geq \int_{E_n} f d\mu \geq \frac{1}{n} \int_{E_n} 1 d\mu = \frac{1}{n} \mu(E_n) > 0,$$

which is impossible! Thus $\int_X f d\mu = 0$ implies $f = 0$ a.e.. $\square$

Corollary

Corollary

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $(f_n)_{n\in\mathbb{N}} \subseteq L^0_+(X)$ and $f_n(x)$ increases to $f(x)$ for a.e. $x \in X$, then

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

Proof. If $f_n(x)$ increases to $f(x)$ for $x \in E$, where $\mu(E^c) = 0$, then $f - f\mathbb{1}_E = 0$ a.e. and $f_n - f_n\mathbb{1}_E = 0$ a.e. By the **(MCT)** we have

$$\int_X f d\mu = \int_X f\mathbb{1}_E d\mu = \lim_{n\to\infty} \int_X f_n\mathbb{1}_E d\mu = \lim_{n\to\infty} \int_X f_n d\mu. \qquad \square$$

Fatou's lemma

Lemma (Fatou's Lemma)

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $(f_n)_{n\in\mathbb{N}} \subseteq L^0_+(X)$, then

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu.$$

Proof. For each $k \geq 0$ we have

$$\inf_{n\geq k} f_n \leq f_j \quad \text{for all} \quad j \geq k.$$

thus

$$\int_X \inf_{n\geq k} f_n d\mu \leq \int_X f_j d\mu \quad \text{for} \quad j \geq k,$$

hence

$$\int_X \inf_{n\geq k} f_n d\mu \leq \inf_{j\geq k} \int_X f_j d\mu.$$

Letting $k \to \infty$ by the **(MCT)** we obtain the claim. $\square$

Corollaries

Corollary

Let $(X, \mathcal{M}, \mu)$ *be a measure space. If* $(f_n)_{n\in\mathbb{N}} \subseteq L^0_+(X)$ *and* $\lim_{n\to\infty} f_n = f$ *a.e., then*

$$\int_X f d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu.$$

Proposition

Let $(X, \mathcal{M}, \mu)$ *be a measure space. If* $f \in L^0_+(X)$ *and* $\int_X f d\mu < \infty$*, then*

$$\mu(\{x \in X : f(x) = \infty\}) = 0$$

and

$$\{x \in X : f(x) > 0\}$$

is σ*-finite.*

Integration of real functions

Let $(X, \mathcal{M}, \mu)$ be a measure space. We know that any measurable $f : X \to \mathbb{R}$ can be written as

$$f = f^+ - f^-$$

If at least one of $\int_X f^+ d\mu$ and $\int_X f^- d\mu$ is finite we define

$$\int_X f(x) d\mu(x) = \int_X f^+(x) d\mu(x) - \int_X f^-(x) d\mu(x) \in [-\infty, \infty].$$

Definition

If $\int_x f^+ d\mu$ and $\int_X f^- d\mu$ are both finite, we say that f is **integrable**.

Remark

Since $|f| = f^+ + f^-$, then f is integrable iff

$$\int_X |f(x)| d\mu(x) < \infty.$$

Useful fact

Proposition

Let $(X, \mathcal{M}, \mu)$ be a measure space. The set of integrable real-valued functions on X is a real vector space and the integral is a linear functional on it.

Proof. The first conclusion follows since $|af + bg| \leq |a||f| + |b||g|$.

- It is easy to check that $\int af = a \int f$ for any $a \in \mathbb{R}$.
- Suppose f, g are integrable, then $\int (f+g) = \int f + \int g$. Indeed, let $h = f + g$, then $h^+ - h^- = f^+ - f^- + g^+ - g^-$ and

$$h^+ + f^- + g^- = h^- + f^+ + g^+,$$

$$\int h^+ + \int f^- + \int g^- = \int h^- + \int g^+ + \int f^+,$$

$$\int h = \int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^- = \int f + \int g. \quad \square$$

Integration of complex functions

Definition

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f : X \to \mathbb{C}$ is measurable function we say that f is **integrable** if

$$\int_X |f(x)| d\mu(x) < \infty.$$

If $E \in \mathcal{M}$, then f is integrable on E if $\int_E |f(x)| d\mu(x) < \infty$.

Remark

Since $|f| \leq |\mathrm{Re}\,(f)| + |\mathrm{Im}\,(f)| \leq 2|f|$. Thus f is integrable iff $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are both integrable. In this case we define

$$\int_X f d\mu = \int_X \mathrm{Re}(f) d\mu + i \int_X \mathrm{Im}(f) d\mu.$$

L^1 space

Definition

Let $(X, \mathcal{M}, \mu)$ be a measure space. The space of all complex-valued integrable functions will be denoted by

$$L^1(X,\mu) = \left\{ f : X \to \mathbb{C} : f \text{ is measurable and } \int_X |f(x)| d\mu(x) < \infty \right\}.$$

We shall abbreviate $L^1(X,\mu)$ to $L^1(X)$ or $L^1(\mu)$ or L^1.

Proposition

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in L^1(X)$, then

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu.$$

Proof

Proof. This is obvious if $\int_X f d\mu = 0$. Assume that $f : X \to \mathbb{R}$, and note

$$\left|\int_X f d\mu\right| = \left|\int_X f^+ d\mu - \int_X f^- d\mu\right| \leq \int_X f^+ d\mu + \int_X f^- d\mu = \int_X |f| d\mu.$$

If $f : X \to \mathbb{C}$ and $\int_X f d\mu \neq 0$, let

$$\alpha = \operatorname{sgn}\left(\int_X f d\mu\right) = \frac{\overline{\int_X f d\mu}}{|\int_X f d\mu|}.$$

Then

$$\left|\int_X f d\mu\right| = \alpha \int_X f d\mu = \int_X \alpha f d\mu \in \mathbb{R},$$

thus $|\int_X f d\mu| = \mathrm{Re}\left(\int_X \alpha f d\mu\right)$ and

$$\left|\int_X f d\mu\right| = \int_X \mathrm{Re}(\alpha f) d\mu \leq \int_X |\mathrm{Re}(\alpha f)| d\mu \leq \int_X |\alpha f| d\mu = \int_X |f| d\mu. \qquad \square$$

Useful facts

Proposition (Chebyshev's inequality)

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in L^1(X)$ and $\lambda > 0$, then

$$\mu(E) \leq \frac{1}{\lambda} \int_E |f(x)| d\mu(x),$$

where $E = \{x \in X : |f(x)| > \lambda\}$.

Proposition

Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) *If $f \in L^1(X)$, then $\{x \in X : f(x) \neq 0\}$ is σ-finite.*

(b) *If $f, g \in L^1(X)$, then $|f - g| = 0$ a.e. iff $f - g = 0$ a.e. iff*

$$\int_E f d\mu = \int_E g d\mu \quad \text{for all} \quad E \in \mathcal{M}.$$

Proof

Proof of (a). Note that

$$\{x \in X : f(x) \neq 0\} = \{x \in X : |f(x)| > 0\} = \bigcup_{n \in \mathbb{N}} A_n,$$

where

$$A_n = \{x \in X : |f(x)| \geq n^{-1}\}.$$

Then we have $A_n \subseteq A_{n+1}$ and

$$\mu\left(\{x \in X : |f(x)| > 0\}\right) = \lim_{n \to \infty} \mu(A_n).$$

By Chebyshev's inequality we have

$$\mu(A_n) \leq n \int_{A_n} |f(x)| d\mu(x) \leq n \int_X |f(x)| \mu(x) < \infty.$$

Proof

Proof of (b). It is clear that $\int_X |f-g| d\mu = 0$ iff $f = g$ a.e..

- If $\int_X |f-g| d\mu = 0$, then $\int_E f d\mu = \int_E g d\mu$ for any $E \in \mathcal{M}$, since

$$\left| \int_E f d\mu - \int_E g d\mu \right| \leq \int_E |f-g| d\mu \leq \int_X |f-g| d\mu = 0.$$

- If $\int_E f d\mu = \int_E g d\mu$ for any $E \in \mathcal{M}$. If $u = \mathrm{Re}(f-g)$, $v = \mathrm{Im}(f-g)$ and if it is false that $f - g$ a.e., then at least one of u^+, u^-, v^+, v^- must be nonzero at the set of positive measure. Suppose that

$$E = \{x \in X : u^+(x) > 0\} \in \mathcal{M}$$

has positive measure. Then

$$\mathrm{Re} \left(\int_E f d\mu - \int_E g d\mu \right) = \int_E u^+ d\mu > 0,$$

since $u^- = 0$ on E. Thus we must have $f = g$ a.e.. $\square$

Remarks

Remark

This proposition shows that for the purposes of integration it makes no difference if we alter functions on null sets.

- One can integrate functions f that are only defined on a measurable set E whose complement is null simply by defining f to be zero (or anything else) on E^c.
- In this fashion we can treat $\overline{\mathbb{R}}$-valued functions that are finite a.e. as real-valued functions for the purposes of integration.
- With this in mind, we shall find it more convenient to redefine $L^1(X, \mu)$ to be the set of equivalence classes of a.e.-defined integrable functions on X, where f and g are considered equivalent iff $f = g$ a.e.
- This new $L^1(X, \mu)$ is still a complex vector space (under pointwise a.e. addition and scalar multiplication).

Remarks

Remark

- Although we shall henceforth view $L^1(X,\mu)$ as a space of equivalence classes, we shall still employ the notation "$f \in L^1(X,\mu)$" to mean that f is an a.e.-defined integrable function. This minor abuse of notation is commonly accepted and rarely causes any confusion.
- There are two advantages of this definition
  1. If $\overline{\mu}$ is a completion of μ there is a natural one-to-one correspondence between $L^1(X,\mu)$ and $L^1(X,\overline{\mu})$. We shall identify these spaces.
  2. $L^1(X,\mu)$ is a metric space with distance function
$$\rho(f,g) = \int_X |f-g|d\mu,$$
$$\rho(f,g) = 0 \quad \Longleftrightarrow \quad f = g \text{ a.e.}$$
  3. We shall refer to convergence with respect to this metric as **convergence in** L^1, thus $f_n \xrightarrow[n\to\infty]{} f$ in L^1 iff $\lim_{n\to\infty} \int_X |f_n - f|d\mu = 0$.

Dominated Convergence Theorem (DCT)

Theorem (Dominated Convergence Theorem (DCT))

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}} \subseteq L^1(X)$ be such that

a. $f_n \xrightarrow[n\to\infty]{} f$ *a.e.,*

b. *there exists $g \in L^1(X)$ such that $g \geq 0$ and $|f_n| \leq g$ a.e. for all $n \in \mathbb{N}$.*

Then $f \in L^1(X)$ and

$$\int_X f d\mu = \int_X \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

Proof

Proof. We can assume (perhaps after redefinition on a null set) that the function f is measurable. Note that

$$|f| = \lim_{n\to\infty} |f_n| \leq g \quad \text{a.e.}.$$

Thus $f \in L^1(X)$, since

$$\int_X |f| d\mu \leq \int_X g d\mu < \infty.$$

By taking real and imaginary parts it suffices to assume that f_n and f are real-valued function. Then

$$|f_n| \leq g \quad \Longleftrightarrow \quad -g \leq f_n \leq g \quad \text{a.e.},$$

which yields

$$g + f_n \geq 0 \quad \text{a.e.} \quad \text{and} \quad g - f_n \geq 0 \quad \text{a.e.}$$

Proof

Thus by Fatou's Lemma we obtain

$$\int_X g d\mu + \int_X f d\mu \leq \liminf_{n\to\infty} \int_X (g + f_n) d\mu = \int_X g d\mu + \liminf_{n\to\infty} \int_X f_n d\mu,$$

and

$$\int_X g d\mu - \int_X f d\mu \leq \liminf_{n\to\infty} \int_X (g - f_n) d\mu = \int_X g d\mu - \limsup_{n\to\infty} \int_X f_n d\mu.$$

Thus

$$\int_X f d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu \leq \limsup_{n\to\infty} \int_X f_n d\mu \leq \int_X f d\mu,$$

and consequently

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu. \qquad \square$$

Theorem

Theorem

Let $(X, \mathcal{M}, \mu)$ *be a measure space. Let* $(f_n)_{n\in\mathbb{N}} \subseteq L^1(X)$ *be such that*

$$\sum_{n\in\mathbb{N}} \int_X |f_n| d\mu < \infty.$$

Then the series

$$\sum_{n\in\mathbb{N}} f_n$$

converges a.e. to a function in $L^1(X)$ *and*

$$\int_X \sum_{n\in\mathbb{N}} f_n d\mu = \sum_{n\in\mathbb{N}} \int_X f_n d\mu.$$

Proof

Proof. By the **(MCT)** we have

$$\int_X \sum_{n\in\mathbb{N}} |f_n| d\mu = \sum_{n\in\mathbb{N}} \int_X |f_n| d\mu < \infty.$$

So the function $g = \sum_{j\in\mathbb{N}} |f_j| \in L^1(X)$ thus

$$g(x) = \sum_{j\in\mathbb{N}} |f_j(x)| < \infty \quad \text{a.e.}$$

Thus the series $\sum_{n\in\mathbb{N}} f_n(x)$ converges a.e. For each $N \in \mathbb{N}$ we have

$$\left| \sum_{j=1}^{N} f_j(x) \right| \leq g(x) \quad \text{a.e.}$$

We apply **(DCT)** and we obtain

$$\int_X \sum_{n\in\mathbb{N}} f_n d\mu = \int_X \lim_{N\to\infty} \sum_{n=1}^{N} f_n d\mu = \lim_{N\to\infty} \int_X \sum_{n=1}^{N} f_n d\mu. \qquad \square$$