Lecture 9

Applications of Dominated Convergence Theorem
Lebesgue integral vs Riemann integral

Graduate Real Analysis

Fall Semester

Simple functions are dense in L^1

Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in L^1(X)$ and $\varepsilon > 0$, there is an integrable simple function $\phi = \sum_{j=1}^{n} a_j \mathbb{1}_{A_j}$ such that

$$\int_X |f - \phi| d\mu < \varepsilon.$$

That is, the integrable simple functions are dense in L^1 in the L^1 metric.

- *Furthermore, if μ is a Lebesgue–Stieltjes measure on $\mathbb{R}$ (or Lebesgue measure on $\mathbb{R}^d$), the sets A_j in the definition of ϕ can be taken to be finite unions of open intervals (open rectangles), and also there is a continuous function g that vanishes outside a bounded interval (bounded rectangle) such that*

$$\int_X |f - g| d\mu < \varepsilon.$$

Proof

Proof. We fix $f \in L^1(X)$ and take a sequence $(\phi_n)_{n\in\mathbb{N}}$ of measurable simple functions $|\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$ such that $\phi_n \xrightarrow[n\to\infty]{} f$ a.e.. Then

$$\lim_{n\to\infty} |\phi_n - f| = 0, \quad \text{and} \quad |\phi_n - f| \leq 2|f|.$$

Thus by the **(DCT)**, we obtain

$$\lim_{n\to\infty} \int_X |\phi_n - f| d\mu = 0.$$

If $\phi_k = \sum_{j=1}^n a_j \mathbb{1}_{A_j}$, where the A_j's are disjoint and the a_j's are nonzero, we note that

$$\mu(A_j) = a_j^{-1} \int_{A_j} |\phi_k| d\mu \leq a_j^{-1} \int_X |f| d\mu < \infty.$$

Proof

If $E, F \in \mathcal{M}$ note that $\mu(E \triangle F) = \int_X |\mathbb{1}_E - \mathbb{1}_F| d\mu$ and recall that:

Theorem (Approximations by finite unions of open rectangles)

If $E \in \mathcal{L}(\mathbb{R}^d)$ and $\lambda_d(E) < \infty$, then for every $\varepsilon > 0$ there is a set A that is a finite union of open rectangles such that $\lambda_d(E \triangle A) < \varepsilon$. The same result is true if μ is a Lebesgue–Stieltjes measure on $\mathbb{R}$.

- If μ is a Lebesgue–Stieltjes measure (or Lebesgue measure of $\mathbb{R}^d$) we can approximate $\mathbb{1}_{A_j}$ arbitrarily closely in the L^1 metric by finite sums of $\mathbb{1}_{R_k}$, where the R_k's are open intervals (or open rectangles).
- Finally, if $R_k = \prod_{i=1}^d (a_i, b_i)$ we can approximate $\mathbb{1}_{R_k}$ in the L^1 metric by continuous functions that vanish outside $\prod_{i=1}^d (a_i, b_i)$.
- For instance, if $d = 1$, given $\varepsilon > 0$ take a continuous picewise linear function g such that $\mathbb{1}_{[a+\varepsilon, b-\varepsilon]} \leq g \leq \mathbb{1}_{(a,b)}$. $\square$

Theorem

Theorem

Let $(X, \mathcal{M}, \mu)$ *be a measure space and suppose that* $f : X \times [a, b] \to \mathbb{C}$ $(-\infty < a < b < \infty)$ *and* $f(\cdot, t) \in L^1(X)$ *for each* $t \in [a, b]$. *Let*

$$F(t) = \int_X f(x, t)\, d\mu(x).$$

- *Suppose that there exists* $g \in L^1(X)$ *such that*

$$|f(x,t)| \leq |g(x)| \quad \text{for all} \quad (x, t) \in X \times [a, b].$$

If

$$\lim_{t \to t_0} f(x, t) = f(x, t_0) \quad \text{for every} \quad x \in X,$$

then

$$\lim_{t \to t_0} F(t) = F(t_0).$$

Proof.

Let

$$f_n(x) = f(x, t_n),$$

where $(t_n)_{n\in\mathbb{N}} \subseteq [a, b]$ is converging to $t_0 \in [a, b]$. Note that

$$|f_n(x)| \leq |g(x)| \in L^1(\mu),$$

and

$$\lim_{n\to\infty} f_n(x) = f(x, t_0).$$

Thus by the **(DCT)** we obtain

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} \int_X f_n(x)d\mu(x) = \int_X \lim_{n\to\infty} f_n(x)d\mu(x) = \int_X f(x, t_0)d\mu(x),$$

which proves that

$$\lim_{n\to\infty} F(t_n) = F(t_0)$$

as desired. □

Theorem

Theorem

Let $(X, \mathcal{M}, \mu)$ *be a measure space and suppose that* $f : X \times [a, b] \to \mathbb{C}$ $(-\infty < a < b < \infty)$ *and* $f(\cdot, t) \in L^1(X)$ *for each* $t \in [a, b]$*. Let*

$$F(t) = \int_X f(x, t)\, d\mu(x).$$

- *Suppose that* $\frac{\partial f}{\partial t}$ *exists and there is* $g \in L^1(X)$ *such that*

$$\left|\frac{\partial f}{\partial t}(x, t)\right| \leq |g(x)| \quad \text{for all} \quad (x, t) \in X \times [a, b].$$

Then F *is differentiable and*

$$F'(t) = \frac{\partial F}{\partial t}(x) = \int_X \frac{\partial f}{\partial t}(x, t) d\mu(x).$$

Proof

Suppose that $(t_n)_{n\in\mathbb{N}} \subseteq [a,b]$ is converging to $t_0 \in [a,b]$, then we may write

$$\frac{\partial f}{\partial t}(x,t_0) = \lim_{n\to\infty} h_n(x),$$

where

$$h_n(x) = \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}.$$

It follows that $\frac{\partial f}{\partial t}$ is measurable and by the mean-value theorem one has

$$|h_n(x)| \leq \sup_{t\in[a,b]} \left|\frac{\partial f}{\partial t}(x,t)\right| \leq |g(x)|,$$

so by the **(DCT)** we conclude

$$F'(t_0) = \lim_{n\to\infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} = \lim_{n\to\infty} \int_X h_n(x)d\mu(x) = \int_X \frac{\partial f}{\partial t}(x,t_0)d\mu(x)$$

as claimed. $\square$

Upper and lower Riemann sums

Definition (Partitions)

Let $[a, b]$ be a given interval. By a partition P of $[a, b]$ we mean a finite set of points $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$.

Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function. Corresponding to each partition P of $[a, b]$ we put

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad \text{and} \quad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x).$$

Upper and lower Riemann sums

We define

$$U(P, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}), \quad \text{and} \quad L(P, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}).$$

- We always have that $L(P, f) \leq U(P, f)$.

Upper and lower Riemann integral

Upper and lower Riemann integral

We define the **upper and lower Riemann integrals of f over $[a, b]$** to be

$$L_a^b(f) = \sup_P L(P, f), \quad \text{and} \quad U_a^b(f) = \inf_P U(P, f),$$

where the inf and the sup are taken over all partitions P of $[a, b]$.

Riemann integral of f over $[a, b]$

If the upper and lower integrals are equal, we say that $f : [a, b] \to \mathbb{R}$ is **Riemann integrable** on $[a, b]$, we write $f \in \mathcal{R}([a, b])$ and we denote the common value (which is called Riemann integral of f over $[a, b]$) by

$$\int_a^b f(x)dx = L_a^b(f) = U_a^b(f).$$

Riemann integral is well-defined

Fact

The upper and lower integrals are defined for every bounded function.

Proof. Let

$$m = \inf_{x \in [a,b]} f(x),$$
$$M = \sup_{x \in [a,b]} f(x).$$

Then

$$m \leq f(x) \leq M \quad \text{for all} \quad x \in [a, b].$$

Therefore

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

for every partition P. □

Example

Example

There is a bounded function $f \in L^1([0,1])$ which is not Riemann integrable.

Proof. Let us define f on $[0,1]$ to be

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \notin \mathbb{Q}. \end{cases}$$

We immediately see that $f \in L^1(\mathbb{R})$. Let us recall

Fact (*)

In any interval $[c,d]$ such that $c < d$ there is a rational and irrational number.

Proof

By the fact $(*)$, for every partition P of $[0,1]$, we have

$$U(P,f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1}) = 1,$$

$$L(P,f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1}) = 0.$$

Therefore

$$L_a^b(f) = \sup_P L(P,f) = 0, \quad \text{and} \quad U_a^b(f) = \inf_P U(P,f) = 1.$$

Hence $L_a^b(f) \neq U_a^b(f)$ and f is not Riemann integrable. $\square$

Theorem

Theorem

If P^* is a refinement of P, i.e. $P^* \supseteq P$, then

$$L(P, f) \leq L(P^*, f),$$
$$U(P^*, f) \leq U(P, f).$$

Proof. We prove the first statement.

- Suppose first that P^* contains just one point more than P. Let this extra point be x^*, and suppose

$$x_{i-1} \leq x^* \leq x_i \quad \text{for some} \quad i \in \{1, 2, \ldots, n\}.$$

- Let

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x),$$

$$w_1 = \inf_{x \in [x_{i-1}, x^*]} f(x), \quad \text{and} \quad w_2 = \inf_{x \in [x^*, x_i]} f(x)$$

Proof

- Then $w_1 \geq m_i$ and $w_2 \geq m_i$ and consequently

$$L(P^*, f) - L(P, f) = w_1(x^* - x_{i-1}) + w_2(x_i - x^*) - m_i(x_i - x_{i-1})$$
$$= (w_1 - m_i)(x^* - x_{i-1}) + (w_2 - m_i)(x_i - x^*) \geq 0.$$

- Finally, if P^* contains k points more than P, we repeat this reasoning k times. The proof of the second statement is analogous. $\square$

Claim (*)

For two partitions P_1, P_2 of an interval $[a, b]$ one has

$$L(P_1, f) \leq U(P_2, f).$$

Proof. Let $P^* = P_1 \cup P_2$ be the common refinement of two partitions P_1 and P_2. By the previous theorem

$$L(P_1, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P_2, f).$$

Theorem

Theorem

For any bounded function $f : [a, b] \to \mathbb{R}$ we have

$$L_a^b(f) \leq U_a^b(f).$$

Proof. By the Claim (*) for two partitions P_1, P_2 of an interval $[a, b]$ one has

$$L(P_1, f) \leq U(P_2, f).$$

Then

$$L_a^b(f) = \sup_{P_1} L(P_1, f) \leq \inf_{P_2} U(P_2, f) = U_a^b(f)$$

This completes the proof of the theorem. □

Theorem

Theorem

A function $f \in \mathcal{R}([a, b])$ if and only if the following condition $(\mathcal{R})$ holds:

- For every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon. \qquad (\mathcal{R})$$

Proof. By the previous theorem, for every partition P we have

$$L(P, f) \leq L_a^b(f) \leq U_a^b(f) \leq U(P, f).$$

Thus the condition $(\mathcal{R})$ implies

$$0 \leq U_a^b(f) - L_a^b(f) \leq U(P, f) - L(P, f) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary $U_a^b(f) = L_a^b(f)$, hence $f \in \mathcal{R}([a, b])$.

Proof

Conversely, suppose that $f \in \mathcal{R}([a,b])$. Then for every $\varepsilon > 0$ there are partitions P_1 and P_2 such that

$$L_a^b(f) - L(P_1, f) < \frac{\varepsilon}{2} \quad \text{and} \quad U(P_2, f) - U_a^b(f) < \frac{\varepsilon}{2}.$$

We choose P to be the **common refinement** of P_1 and P_2. Then

$$\begin{gathered} U(P,f) \leq U(P_2, f) \\ \leq U_a^b(f) + \frac{\varepsilon}{2} = \int_a^b f(x)dx + \frac{\varepsilon}{2} = L_a^b(f) + \frac{\varepsilon}{2} \\ \leq L(P_1, f) + \varepsilon \leq L(P, f) + \varepsilon. \end{gathered}$$

This proves condition $(\mathcal{R})$ and completes the proof of the theorem. $\square$

Theorem

Theorem (**)

If condition $(\mathcal{R})$ holds for $P = \{x_0, \ldots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|(x_i - x_{i-1}) < \varepsilon.$$

Proof. Note that $f(s_i), f(t_i)$ lies in $[m_i, M_i]$, hence by the triangle inequality $|f(t_i) - f(s_i)| \leq \underbrace{M_i - m_i}_{\text{length}}$. Hence

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|(x_i - x_{i-1}) \leq \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$$
$$= U(P, f) - L(P, f) < \varepsilon.$$

This completes the proof. □

Theorem

Theorem

If $f \in \mathcal{R}([a,b])$ and the hypotheses of $(**)$ hold, then

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - \int_a^b f(x)\,dx \right| < \varepsilon.$$

Proof. It is enough to note that

$$L(P,f) \le \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \le U(P,f),$$

$$L(P,f) \le \int_a^b f(x)dx \le U(P,f). \qquad \square$$

Lebesgue integral vs Riemann integral

Theorem

Let $f : [a,b] \to \mathbb{R}$ be a bounded function . If f is Riemann integrable then f is Lebesgue measurable and integrable on $[a,b]$ and

$$\underbrace{\int_a^b f(x)dx}_{Riemann} = \underbrace{\int_{[a,b]} f(x)d\lambda(x)}_{Lebesgue}.$$

Proof. Assume that $f \in \mathcal{R}([a,b])$. For each partition $P = (t_j)_{j=0}^n$ consider

$$G_P = \sum_{j=1}^n M_j \mathbb{1}_{[t_{j-1},t_j]}, \quad \text{and} \quad g_P = \sum_{j=1}^n m_j \mathbb{1}_{[t_{j-1},t_j]}.$$

Then

$$U(P,f) = \int_{[a,b]} G_P(x)d\lambda(x), \quad \text{and} \quad L(P,f) = \int_{[a,b]} g_P(x)d\lambda(x).$$

Proof

By the Riemann integrability there is a sequence $(P_k)_{k\in\mathbb{N}}$ of partitions of $[a,b]$ such that $P_k = \{t_{k,0},\ldots,t_{k,n_k}\}$ such that

$$a = t_{k,0} < t_{k,1} < \ldots < t_{k,n_k-1} < t_{k,n_k} = b$$

and $P_k \subset P_{k+1}$ for any $k \in \mathbb{N}$. Moreover,

$$\lim_{k\to\infty} \max_{1\leq j\leq n_k} (t_{k,j} - t_{k,j-1}) = 0,$$

and

$$\lim_{k\to\infty} U(P_k,f) = \lim_{k\to\infty} L(P_k,f) = \int_a^b f(x)dx.$$

We also see that g_{P_k} increases and G_{P_k} decreases as $k\to\infty$, thus we may write

$$G = \lim_{k\to\infty} G_{P_k}, \quad \text{and} \quad g = \lim_{k\to\infty} g_{P_k}.$$

Proof

Then $g \leq f \leq G$ and by the **(DCT)**, we conclude

$$\int_{[a,b]} G(x)d\lambda(x) = \int_{[a,b]} g(x)d\lambda(x) = \int_a^b f(x)dx.$$

Hence

$$\int_{[a,b]} (G(x) - g(x))d\lambda(x) = 0,$$

so $G = g$ a.e., and consequently $G = g = f$ a.e..

But G is measurable (as a limit of a sequence of simple functions) and λ is complete, thus f is measurable and

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Riemann}} = \int_{[a,b]} G(x)d\lambda(x) = \underbrace{\int_{[a,b]} f(x)d\lambda(x)}_{\text{Lebesgue}}. \qquad \square$$

Characterization of Riemann integrability

Theorem

A bounded function $f \in \mathcal{R}([a,b])$ iff f is a.e. continuous.

Proof of ($\Longrightarrow$). Assume that $f \in \mathcal{R}([a,b])$. For $x \in [a,b]$ define

$$g(x) = \sup_{\delta>0} \inf_{y\in[a,b]\cap B(x,\delta)} f(y),$$
$$h(x) = \inf_{\delta>0} \sup_{y\in[a,b]\cap B(x,\delta)} f(y),$$

where $B(x,\delta) = (x-\delta, x+\delta)$. It is not difficult to see that $g \leq f \leq h$ and f is continuous at $x \in [a,b]$ iff $g(x) = h(x)$. Moreover, for any partition P of $[a,b]$ we have

$$L(P,g) \leq L(P,f) \leq L(P,h),$$
$$U(P,g) \leq U(P,f) \leq U(P,h).$$

Proof

In fact, we have $U(P,f) = U(P,h)$ and $L(P,g) = L(P,f)$, since for any open interval $(c,d) \subset [a,b]$ we have

$$\inf_{x\in(c,d)} g(x) = \inf_{x\in(c,d)} f(x), \quad \text{and} \quad \sup_{x\in(c,d)} h(x) = \sup_{x\in(c,d)} f(x).$$

Indeed, for any $z \in (c,d)$ and $B(z,\delta) \subseteq (c,d)$ we have

$$\inf_{x\in(c,d)} g(x) \le \inf_{x\in(c,d)} f(x) \le \inf_{y\in B(z,\delta)} f(y) \le g(z).$$

Since $z \in (c,d)$ is arbitrary we obtain $\inf_{x\in(c,d)} g(x) = \inf_{x\in(c,d)} f(x)$. Similarly, we show that $\sup_{x\in(c,d)} h(x) = \sup_{x\in(c,d)} f(x)$. It follows that

$$L_a^b(f) = L_a^b(g) \le U_a^b(g) \le U_a^b(f),$$
$$L_a^b(f) \le L_a^b(h) \le U_a^b(h) = U_a^b(f),$$

and consequently $g, h \in \mathcal{R}([a,b])$, since $f \in \mathcal{R}([a,b])$.

Proof

Since $g \leq h$ and $\int_{[a,b]}(h-g)d\lambda = 0$ we obtain that $g = h$ a.e., and consequently $g = h = f$ a.e., thus f is continuous a.e. as we claimed.

Proof of ($\Longleftarrow$). Assume that f is continuous a.e., and for $n \in \mathbb{N}$ let P_n be the dissection of $[a, b]$ into 2^n equal points. Set

$$h_n(x) = \sup_{y \in (c,d)} f(y), \quad \text{and} \quad g_n(x) = \inf_{y \in (c,d)} f(y),$$

where (c, d) is an open interval containing x, whose endpoints are two consecutive elements from P_n, and set $h_n(x) = g_n(x) = f(x)$ if $x \in P_n$.

Then we see that $(g_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ are respectively, increasing and decreasing sequences of functions, each function constant on each of a finite family of intervals covering $[a, b]$, and

$$U(P_n, f) = \int_{[a,b]} h_n(x) d\lambda(x), \quad \text{and} \quad L(P_n, f) = \int_{[a,b]} g_n(x) d\lambda(x).$$

Proof

We also see that

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} h_n(x) = f(x)$$

at any point $x \in [a, b]$ at which f is continuous. By the **(DCT)**, we conclude that

$$\lim_{n\to\infty} \int_{[a,b]} g_n(x) d\lambda(x) = \int_{[a,b]} f(x) d\lambda(x) = \lim_{n\to\infty} \int_{[a,b]} h_n(x) d\lambda(x),$$

but this means that

$$L_a^b(f) \geq \int_{[a,b]} f(x) d\lambda(x) \geq U_a^b(f),$$

so these are all equal and f is Riemann integrable. $\square$