Lecture 12

The Fourier transform

MATH 503, FALL 2025

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Functions with moderate decrease

Definition

For each $a > 0$ we denote by $\mathfrak{F}_a$ the class of all functions f that satisfy the following two conditions:

(i) The function f is holomorphic in the horizontal strip

$$S_a = \{z \in \mathbb{C} : |\operatorname{Im}(z)| < a\}.$$

(ii) There exists a constant $A > 0$ such that

$$|f(x+iy)| \leq \frac{A}{1+x^2} \quad \text{for all} \quad x \in \mathbb{R} \quad \text{and} \quad |y| < a.$$

In other words, $\mathfrak{F}_a$ consists of those holomorphic functions on S_a that are of moderate decay on each horizontal line $\operatorname{Im}(z) = y$, uniformly for all $y \in (-a, a)$.

Functions with moderate decrease

Example

- For example,

$$f(z) = e^{-\pi z^2}$$

  belongs to $\mathfrak{F}_a$ for all $a > 0$.

- Also, the function

$$f(z) = \frac{1}{\pi} \frac{c}{c^2 + z^2}$$

  which has simple poles at $z = \pm ci$, belongs to $\mathfrak{F}_a$ for all $0 < a < c$.

- Another example is provided by

$$f(z) = 1/\cosh \pi z,$$

  which belongs to $\mathfrak{F}_a$ whenever $|a| < 1/2$.

Functions with moderate decrease

Remarks

- Note also that a simple application of the Cauchy integral formula shows that if $f \in \mathfrak{F}_a$, then for every $n \in \mathbb{N}$, the n^{th} derivative of f belongs to $\mathfrak{F}_b$ for all b with $0 < b < a$. It is a simple exercise.
- Finally, we denote by $\mathfrak{F}$ the class of all functions that belong to $\mathfrak{F}_a$ for some $a > 0$. In other words, we can write $\mathfrak{F} = \bigcup_{a>0} \mathfrak{F}_a$.
- The condition of moderate decrease can be weakened somewhat by replacing the order of decrease of

$$\frac{A}{1+x^2} \quad \text{by} \quad \frac{A}{1+|x|^{1+\varepsilon}}$$

for any $\varepsilon > 0$. One can observe that many of the results below remain unchanged with this less restrictive condition.

Fourier transform

Theorem

If f belongs to the class $\mathfrak{F}_a$ for some $a > 0$, then

$$|\hat{f}(\xi)| \leq Be^{-2\pi b|\xi|},$$

for any $0 \leq b < a$, where

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)dx.$$

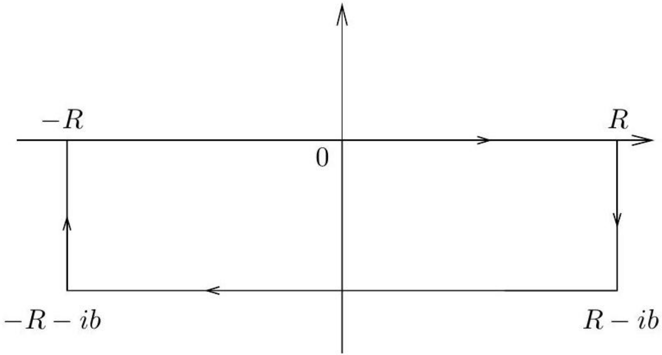
Proof: The case $b = 0$ simply says that $\hat{f}$ is bounded. Indeed, we have

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}} |f(x)|dx \leq \int_{\mathbb{R}} \frac{A}{1+x^2}dx.$$

Hence we can take $B = A\pi$ and we are done.

Fourier transform

- Now suppose $0 < b < a$ and assume first that $\xi > 0$. The main step consists of shifting the contour of integration, that is the real line, down by b.
- More precisely, consider the function $g(z) = f(z)e^{-2\pi iz\xi}$ as well as the contour



Fourier transform

- We claim that as R tends to infinity, the integrals of g over the two vertical sides converge to zero.
- For example, the integral over the vertical segment on the left can be estimated by

$$\begin{aligned}\left|\int_{-R-ib}^{-R} g(z)dz\right| &\le \int_0^b \left|f(-R-it)e^{-2\pi i(-R-it)\xi}\right| dt \\ &\le \int_0^b \frac{A}{R^2} e^{-2\pi t\xi} dt \\ &= O\left(1/R^2\right).\end{aligned}$$

- Thus

$$\lim_{R\to\infty}\left|\int_{-R-ib}^{-R} g(z)dz\right| = 0.$$

- A similar estimate for the other side proves our claim.

Fourier transform

- Therefore, by Cauchy's theorem applied to the large rectangle, we find in the limit as R tends to infinity that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x - ib) e^{-2\pi i(x-ib)\xi} dx,$$

which leads to the estimate

$$|\hat{f}(\xi)| \leq \int_{-\infty}^{\infty} \frac{A}{1+x^2} e^{-2\pi b\xi} dx \leq Be^{-2\pi b\xi},$$

where $B = A\pi$.

- A similar argument for $\xi < 0$, but this time shifting the real line up by b, allows us to finish the proof of the theorem. $\square$

Fourier inversion formula

Remark

- The previous result says that whenever $f \in \mathfrak{F}$, then $\hat{f}$ has rapid decay at infinity.
- We remark that the further we can extend f (that is, the larger a), then the larger we can choose b, hence the better the decay.
- It is possible to describe those f for which $\hat{f}$ has the ultimate decay condition: compact support.

Theorem

If $f \in \mathfrak{F}$, then the Fourier inversion holds, namely

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \quad \textit{for all} \quad x \in \mathbb{R}.$$

Fourier inversion formula

Lemma

If $A > 0$ and $B \in \mathbb{R}$, then

$$\int_0^\infty e^{-(A+iB)\xi} d\xi = \frac{1}{A+iB}.$$

Proof: Since $A > 0$ and $B \in \mathbb{R}$, we have $\left|e^{-(A+iB)\xi}\right| = e^{-A\xi}$, and the integral converges. By definition

$$\int_0^\infty e^{-(A+iB)\xi} d\xi = \lim_{R\to\infty} \int_0^R e^{-(A+iB)\xi} d\xi.$$

However,

$$\int_0^R e^{-(A+iB)\xi} d\xi = \left[-\frac{e^{-(A+iB)\xi}}{A+iB}\right]_0^R,$$

which tends to $1/(A+iB)$ as R tends to infinity. □

Fourier inversion formula

Proof: We can now prove the inversion theorem.

- Once again, the sign of ξ matters, so we begin by writing

$$\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi = \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i x\xi} d\xi + \int_{0}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi.$$

- For the second integral we argue as follows. Say $f \in \mathfrak{F}_a$ and choose $0 < b < a$. Arguing as the proof of the previous theorem (changing the contour of integration), we get

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(u - ib) e^{-2\pi i (u-ib)\xi} du,$$

so that with an application of the lemma and the convergence of the integration in ξ, we find

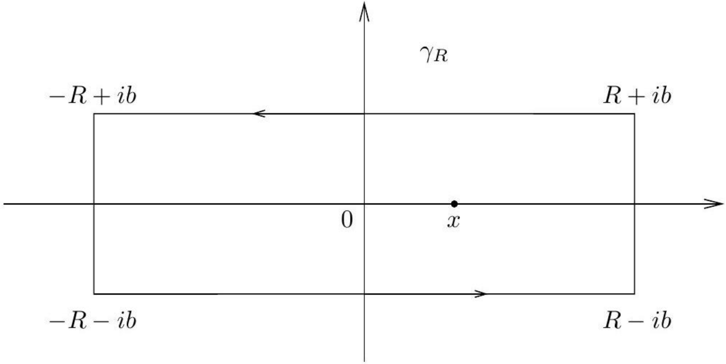
Fourier inversion formula

$$
\begin{aligned}
\int_0^\infty \hat{f}(\xi) e^{2\pi i x\xi} d\xi &= \int_0^\infty \int_{-\infty}^\infty f(u-ib) e^{-2\pi i(u-ib)\xi} e^{2\pi i x\xi} du d\xi \\
&= \int_{-\infty}^\infty f(u-ib) \int_0^\infty e^{-2\pi i(u-ib-x)\xi} d\xi du \\
&= \int_{-\infty}^\infty f(u-ib) \frac{1}{2\pi b + 2\pi i(u-x)} du \\
&= \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{f(u-ib)}{u-ib-x} du \\
&= \frac{1}{2\pi i} \int_{L_1} \frac{f(\zeta)}{\zeta - x} d\zeta,
\end{aligned}
$$

where L_1 denotes the line $\{u - ib : u \in \mathbb{R}\}$ traversed from left to right. (In other words, L_1 is the real line shifted down by b.)

Fourier inversion formula

- For the integral when $\xi < 0$, a similar calculation gives
$$\int_{-\infty}^{0} \hat{f}(\xi)e^{2\pi i x\xi}d\xi = -\frac{1}{2\pi i}\int_{L_2}\frac{f(\zeta)}{\zeta - x}d\zeta,$$
where L_2 is the real line shifted up by b, with orientation from left to right. Now given $x \in \mathbb{R}$, consider the contour γ_R:



Fourier inversion formula

- The function $f(\zeta)/(\zeta - x)$ has a simple pole at x with residue $f(x)$, so the residue formula gives
$$f(x) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(\zeta)}{\zeta - x} d\zeta.$$
- Letting R tend to infinity, one checks easily that the integral over the vertical sides goes to 0 and therefore, combining with the previous results, we get
$$\begin{aligned} f(x) &= \frac{1}{2\pi i}\int_{L_1} \frac{f(\zeta)}{\zeta - x} d\zeta - \frac{1}{2\pi i}\int_{L_2} \frac{f(\zeta)}{\zeta - x} d\zeta \\ &= \int_0^\infty \hat{f}(\xi) e^{2\pi i x\xi} d\xi + \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i x\xi} d\xi \\ &= \int_{-\infty}^\infty \hat{f}(\xi) e^{2\pi i x\xi} d\xi, \end{aligned}$$

and the theorem is proved. □

Poisson summation formula

Theorem

If $f \in \mathfrak{F}$, then

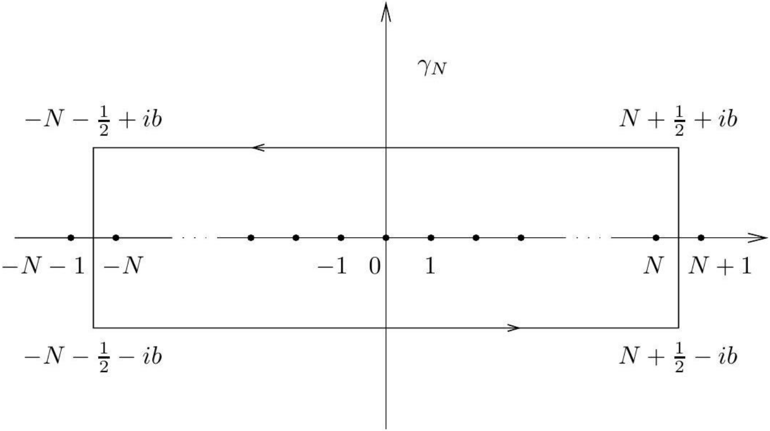
$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Proof: Say $f \in \mathfrak{F}_a$ and choose some b satisfying $0 < b < a$.

- The function
$$\frac{1}{e^{2\pi iz} - 1}$$
has simple poles with residue $1/(2\pi i)$ at the integers.
- Thus
$$\frac{f(z)}{e^{2\pi iz} - 1}$$
has simple poles at the integers $n \in \mathbb{Z}$, with residues $f(n)/2\pi i$.

Poisson summation formula

- We may therefore apply the residue formula to the contour γ_N:



where N is an integer.

Poisson summation formula

- This yields
$$\sum_{|n|\leq N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi iz}-1}dz.$$
Letting N tend to infinity, and recalling that f has moderate decrease, we see that the sum converges to
$$\sum_{n\in\mathbb{Z}} f(n).$$
- Also that the integral over the vertical segments goes to 0.
- Therefore, in the limit we get
$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} \frac{f(z)}{e^{2\pi iz}-1}dz - \int_{L_2} \frac{f(z)}{e^{2\pi iz}-1}dz, \tag{*}$$
where L_1 and L_2 are the real line shifted down and up by b, respectively.

Poisson summation formula

- Now we use the fact that if $|w| > 1$, then

$$\frac{1}{w-1} = w^{-1} \sum_{n=0}^{\infty} w^{-n}$$

to see that on L_1 (where $\left|e^{2\pi iz}\right| > 1$) we have

$$\frac{1}{e^{2\pi iz}-1} = e^{-2\pi iz} \sum_{n=0}^{\infty} e^{-2\pi inz}.$$

- Also if $|w| < 1$, then

$$\frac{1}{w-1} = -\sum_{n=0}^{\infty} w^{n}$$

so that on L_2 we have

$$\frac{1}{e^{2\pi iz}-1} = -\sum_{n=0}^{\infty} e^{2\pi inz}.$$

Poisson summation formula

- Substituting these observations in (*) we find that

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} f(z)\left(e^{-2\pi iz}\sum_{n=0}^{\infty} e^{-2\pi inz}\right) dz + \int_{L_2} f(z)\left(\sum_{n=0}^{\infty} e^{2\pi inz}\right) dz$$
$$= \sum_{n=0}^{\infty}\int_{L_1} f(z)e^{-2\pi i(n+1)z}dz + \sum_{n=0}^{\infty}\int_{L_2} f(z)e^{2\pi inz}dz$$
$$= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-2\pi i(n+1)x}dx + \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} f(x)e^{2\pi inx}dz$$
$$= \sum_{n=0}^{\infty}\hat{f}(n+1) + \sum_{n=0}^{\infty}\hat{f}(-n) = \sum_{n\in\mathbb{Z}}\hat{f}(n),$$

where we have shifted L_1 and L_2 back to the real line according to equation and its analogue for the shift down. □

Theta function

- First, we recall that the function $e^{-\pi x^2}$ was its own Fourier transform:

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx = e^{-\pi \xi^2}.$$

- For fixed values of $t > 0$ and $a \in \mathbb{R}$, the change of variables

$$x \mapsto t^{1/2}(x + a)$$

in the above integral shows that the Fourier transform of the function $f(x) = e^{-\pi t(x+a)^2}$ is

$$\hat{f}(\xi) = t^{-1/2} e^{-\pi \xi^2/t} e^{2\pi i a\xi}.$$

- Applying the Poisson summation formula to the pair f and $\hat{f}$ (which belong to $\mathfrak{F}$) provides the following relation:

$$\sum_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} t^{-1/2} e^{-\pi n^2/t} e^{2\pi i n a}. \qquad (**)$$

Theta function

- This identity has noteworthy consequences. For instance, the special case $a = 0$ is the transformation law for a version of the **"theta function"**: if we define ϑ for $t > 0$ by the series

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t},$$

then the relation (**) says precisely that

$$\vartheta(t) = t^{-1/2}\vartheta(1/t) \quad \text{for} \quad t > 0.$$

- This equation will be used to derive the key functional equation of the Riemann zeta function, and this leads to its analytic continuation.

Example

- For another application of the Poisson summation formula we recall that the function $1/\cosh \pi x$ was also its own Fourier transform:

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh \pi x} dx = \frac{1}{\cosh \pi\xi}.$$

- This implies that if $t > 0$ and $a \in \mathbb{R}$, then the Fourier transform of the function

$$f(x) = e^{-2\pi i a x}/\cosh(\pi x/t),$$

is

$$\hat{f}(\xi) = t/\cosh(\pi(\xi + a)t),$$

and the Poisson summation formula yields

$$\sum_{n=-\infty}^{\infty} \frac{e^{-2\pi i a n}}{\cosh(\pi n/t)} = \sum_{n=-\infty}^{\infty} \frac{t}{\cosh(\pi(n+a)t)}.$$

Paley–Wiener type theorem

Theorem

Suppose $\hat{f}$ satisfies the decay condition

$$|\hat{f}(\xi)| \leq Ae^{-2\pi a|\xi|}$$

for some constants $a, A > 0$. Then $f(x)$ is the restriction to $\mathbb{R}$ of a function $f(z)$ holomorphic in the strip

$$S_b = \{z \in \mathbb{C} : |\operatorname{Im}(z)| < b\},$$

for any $0 < b < a$.

Proof: Define

$$f_n(z) = \int_{-n}^{n} \hat{f}(\xi) e^{2\pi i \xi z} d\xi$$

and note that f_n is entire. (Why?)

Paley–Wiener type theorem

- Observe also that $f(z)$ may be defined for all z in the strip S_b by

$$f(z) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi z} d\xi,$$

because the integral converges absolutely by our assumption on $\hat{f}$: it is majorized by

$$A \int_{-\infty}^{\infty} e^{-2\pi a|\xi|} e^{2\pi b|\xi|} d\xi,$$

which is finite if $b < a$.

- Moreover, for $z \in S_b$, we have

$$|f(z) - f_n(z)| \leq A \int_{|\xi| \geq n} e^{-2\pi a|\xi|} e^{2\pi b|\xi|} d\xi \xrightarrow[n\to\infty]{} 0,$$

and thus the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly in S_b, which, proves the theorem. (Why?) □