# Lecture 14

## The Riemann mapping theorem

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# The Riemann mapping theorem

- $D = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc centered at the origin.

**Theorem**

*Let $\Omega \neq \mathbb{C}$ be a simply connected region. Then $\Omega$ is conformally equivalent to $D$. Moreover, the assumption $\Omega \neq \mathbb{C}$ is necessary.*

**Remark**

- In view of the Liouville theorem, the assumption $\Omega \neq \mathbb{C}$ is necessary. Indeed, if $f : \mathbb{C} \to D$ is a conformal map, then $f$ is bounded, since $|f(z)| < 1$ for all $z \in \mathbb{C}$. Hence, by the Liouville theorem $f$ must be constant, but then it cannot be injective.
- However, $\mathbb{C}$ and $D$ are homeomorphic. The mapping

$$\mathbb{C} \ni z \mapsto \frac{z}{1 + |z|} \in D$$

is the desired homeomorphism.

# Simple topological lemma

**Lemma**

*Let $\Omega$ be an open subset of $\mathbb{C}$. Then there exists a sequence $(K_n)_{n\in\mathbb{N}} \subseteq \Omega$ of compact sets such that*

$$\Omega = \bigcup_{n=1}^{\infty} K_n \quad \textit{and} \quad K_n \subseteq \operatorname{int} K_{n+1} \quad \textit{for} \quad n \in \mathbb{N},$$

*where* $\operatorname{int} K_{n+1}$ *denotes the interior of $K_{n+1}$. Further for a compact set $K \subseteq \Omega$, we have $K \subseteq K_n$ for some $n \in \mathbb{N}$.*

**Proof:** For $n \in \mathbb{N}$, let

$$K_n = \overline{D}(0, n) \cap \{z \in \Omega : d(z, \mathbb{C} \setminus \Omega) \geq 1/n\}.$$

- It is clear that $K_n \subseteq \Omega$ and it is bounded.
- Each $K_n$ is closed and consequently compact.
- Moreover, $K_n \subseteq K_{n+1}$ for $n \in \mathbb{N}$.

# Simple topological lemma

- We prove that $K_n \subseteq \operatorname{int} K_{n+1}$. Taking $r = \frac{1}{n} - \frac{1}{n+1}$, it suffices to show
$$D(z,r) \subseteq K_{n+1} \quad \text{for} \quad z \in K_n.$$
- Let $z \in K_n$ and $\zeta \in D(z,r)$. Then
$$|\zeta - z| < \frac{1}{n} - \frac{1}{n+1} \quad \text{and} \quad |z-a| \geq \frac{1}{n} \quad \text{for} \quad a \notin \Omega.$$
- Therefore for $a \notin \Omega$, we have
$$|\zeta - a| \geq |z-a| - |\zeta - z| \geq \frac{1}{n} - \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{n+1}.$$
- This implies $\zeta \in K_{n+1}$, since $|\zeta| < |z| + \frac{1}{n} \leq n + \frac{1}{n} \leq n+1$.
- Hence $K_n \subseteq \operatorname{int} K_{n+1}$ for $n \in \mathbb{N}$ and $K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots$.
- We observe that $\bigcup_{n=1}^{\infty} K_n \subseteq \Omega$, since each $K_n \subseteq \Omega$.
- For the reverse inclusion, let $z \in \Omega$ and $M \in \mathbb{N}$ be so that $|z| \leq M$. Since $\Omega$ is open, there exists $\rho > 0$ such that $D(z,\rho) \subseteq \Omega$. Let $N \in \mathbb{N}$ be the least integer greater than or equal to $\max(M, \rho^{-1})$.

# Simple topological lemma

- Then $|z| \leq M \leq N$ and for $a \notin \Omega$ we have $|z-a| \geq \rho \geq \frac{1}{N}$. Therefore $z \in K_N$ and thus $\Omega \subseteq \bigcup_{n=1}^{\infty} K_n$, hence $\Omega = \bigcup_{n=1}^{\infty} K_n$.
- We also have a stronger result $\Omega = \bigcup_{n=1}^{\infty} \operatorname{int} K_n$.
- Indeed, since $K_n \subseteq \operatorname{int} K_{n+1}$, we have

$$\Omega \subseteq \bigcup_{n=1}^{\infty} K_n \subseteq \bigcup_{n=2}^{\infty} \operatorname{int} K_n \subseteq \bigcup_{n=1}^{\infty} \operatorname{int} K_n \subseteq \Omega.$$

- Let $K$ be a compact subset of $\Omega$. Then

$$K \subseteq \bigcup_{n=1}^{\infty} \operatorname{int} K_n$$

is an open cover of $K$. Therefore, $K \subseteq \bigcup_{n=1}^{P} \operatorname{int} K_n$ for some $P \in \mathbb{N}$.
- Thus

$$K \subseteq K_1 \cup K_2 \cup \ldots \cup K_P = K_P. \qquad \square$$

# Hurwitz lemma

**Lemma**

*Let $\Omega \subseteq \mathbb{C}$ be a region and $(f_n)_{n\in\mathbb{N}}$ be a sequence such that $f_n \in H(\Omega)$ converges uniformly on compact subsets of $\Omega$. Assume that $f_n$ is one-to-one for any $n \in \mathbb{N}$. Then the limit function $f = \lim_{n\to\infty} f_n$ is either constant or one-to-one on $\Omega$.*

**Proof:** Note that $f \in H(\Omega)$. (Why?)

- Assume that $f$ is not one-to-one, then there exist $z_1, z_2 \in \Omega$ such that $z_1 \neq z_2$ with $f(z_1) = f(z_2)$.
- For $n \in \mathbb{N}$, we define $g_n \in H(\Omega)$, and $g \in H(\Omega)$ by setting

$$g_n(z) = f_n(z) - f_n(z_1), \quad \text{and} \quad g(z) = f(z) - f(z_1).$$

  Then $g(z_2) = 0$ and also $g_n(z_2) \neq 0$, since $f_n$ is one-to-one.
- Further, we may suppose that $g$ is not constant in $\Omega$ otherwise $f$ is constant in $\Omega$ and the assertion follows.

# Hurwitz lemma

- Since the zeros of $g$ are isolated, there is $r > 0$ so that $|z_1 - z_2| > r$ and $g(z) \neq 0$ whenever $z \in \Omega$ satisfies $0 < |z - z_2| \leq r$.
- Let $\gamma$ be a circle centered at $z_2$ with radius $r$. Then there exists $\delta > 0$ such that $|g(z)| > \delta$ for $z \in \gamma^*$.
- By the uniform convergence there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we have

$$|g_n(z) - g(z)| < \frac{\delta}{2} \quad \text{uniformly for any} \quad z \in \gamma^*.$$

- Therefore for $z \in \gamma^*$, we have

$$|g_n(z)| \geq |g(z)| - |g_n(z) - g(z)| \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

- Now we see that $\frac{1}{g_n(z)}$ converges uniformly to $\frac{1}{g(z)}$ on $\gamma^*$. Also $g_n'(z)$ converges uniformly to $g'(z)$ on $\gamma^*$.

# Hurwitz lemma

- Hence
$$\lim_{n\to\infty}\frac{g_n'(z)}{g_n(z)}=\frac{g'(z)}{g(z)}$$
uniformly on $\gamma^*$.
- We observe that $\frac{g_n'(z)}{g_n(z)}\in H\left(\overline{D}\left(z_2,r\right)\right)$. Now by the Cauchy theorem we obtain that
$$0=\frac{1}{2\pi i}\int_\gamma \frac{g_n'(z)}{g_n(z)}dz \quad \text{for} \quad n\in\mathbb{N}.$$
- Consequently, by the argument principle
$$0=\lim_{n\to\infty}\frac{1}{2\pi i}\int_\gamma \frac{g_n'(z)}{g_n(z)}dz=\frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}dz=N_g\geq 1.$$
- This is a contradiction completing the proof of Hurwitz lemma. □

# Normal families of analytic functions

**Definition**

(i) Let $\Omega \subseteq \mathbb{C}$ be a region and $\mathcal{F}$ be a family of analytic functions in $\Omega$. Thus $\mathcal{F}$ is a sub-family of $H(\Omega)$. Further $\mathcal{F}$ is called a **normal family** if every sequence of elements of $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of $\Omega$.

(ii) $\mathcal{F}$ is **uniformly bounded** on compact subsets of $\Omega$ if for every compact subset $K$ of $\Omega$ there exists $M = M(K)$ such that

$$|f(z)| \leq M \quad \text{for} \quad f \in \mathcal{F},\ z \in K.$$

(iii) The family $\mathcal{F}$ is called **equicontinuous** on compact subsets of $\Omega$ if for $\varepsilon > 0$ and compact subset $K \subseteq \Omega$ there exists $\delta > 0$ depending only on $\varepsilon$ and $K$ such that $|f(z_1) - f(z_2)| < \varepsilon$ for $f \in \mathcal{F}$ and $z_1, z_2 \in K$ satisfying $|z_1 - z_2| < \delta$.

# Montel lemma

**Remark**

- The limit of the subsequence in the above definition (i) belongs to $H(\Omega)$ but it **need not belong** to $\mathcal{F}$.

**Lemma**

*Let $\mathcal{F}$ be a family of $H(\Omega)$ with $\Omega \subseteq \mathbb{C}$ region. Assume that $\mathcal{F}$ is uniformly bounded on compact subsets of $\Omega$. Then $\mathcal{F}$ is a normal family.*

**Proof:** We show that $\mathcal{F}$ is equicontinuous on compact subsets of $\Omega$.

- By the topological lemma, there exists a sequence of compact sets $(K_n)_{n\in\mathbb{N}} \subseteq \Omega$ such that $K_n \subseteq \operatorname{int} K_{n+1}$ for $n \in \mathbb{N}$ that satisfies

$$\Omega = \bigcup_{n=1}^{\infty} K_n.$$

Also, every compact subset of $\Omega$ is contained in $K_n$ for some $n \in \mathbb{N}$.

# Montel lemma

- Let $n \in \mathbb{N}$. Since $K_n$ is compact, $(\operatorname{int} K_{n+1})^c$ is closed and

$$K_n \cap (\operatorname{int} K_{n+1})^c = \emptyset,$$

- We can also find $\delta_n > 0$ such that

$$|z_1 - z_2| > 2\delta_n \quad \text{for} \quad z_1 \in K_n, \quad z_2 \notin \operatorname{int} K_{n+1}.$$

- Thus

$$\overline{D}(z, 2\delta_n) \subseteq \operatorname{int} K_{n+1} \subseteq K_{n+1} \quad \text{for} \quad z \in K_n.$$

- Let $z' \in K_n$, and $z'' \in K_n$ with $|z' - z''| < \delta_n$.
- Let $\gamma$ be a circle centered at $z'$ and with radius $2\delta_n$.
- Thus $z''$ lies inside the circle $\gamma$ and $\gamma^* \subseteq K_{n+1}$.

# Montel lemma

- Let $f \in \mathcal{F}$, then by the Cauchy integral formula

$$f\left(z'\right) - f\left(z''\right) = \frac{1}{2\pi i}\int_{\gamma} f(\zeta)\left(\frac{1}{\zeta - z'} - \frac{1}{\zeta - z''}\right) d\zeta$$
$$= \frac{z' - z''}{2\pi i}\int_{\gamma} \frac{f(\zeta)}{\left(\zeta - z'\right)\left(\zeta - z''\right)} d\zeta.$$

- If $|z' - z''| < \delta_n$, we observe that

$$\left|\zeta - z'\right| = 2\delta_n \quad \text{and} \quad \left|\zeta - z''\right| = \left|\zeta - z' + z' - z''\right| > 2\delta_n - \delta_n = \delta_n.$$

- Since $\mathcal{F}$ is uniformly bounded on compact subsets of $\Omega$, there exists a constant $M\left(K_{n+1}\right)$ depending only on $K_{n+1}$ such that

$$\left|f\left(z'\right) - f\left(z''\right)\right| < \frac{|z' - z''|}{2\pi}\frac{4\pi\delta_n}{2\delta_n^2} M\left(K_{n+1}\right) = \frac{M\left(K_{n+1}\right)}{\delta_n}\left|z' - z''\right|.$$

- The above inequality holds for all $f \in \mathcal{F}$ and $z', z'' \in K_n$ whenever $|z' - z''| < \delta_n$.

# Montel lemma

- Let $\varepsilon > 0$ and set
$$\delta = \delta(n) = \frac{\varepsilon \delta_n}{\varepsilon + M\left(K_{n+1}\right)} < \delta_n.$$
- Then for $|z' - z''| < \delta$, we have
$$\frac{M\left(K_{n+1}\right)}{\delta_n}\left|z' - z''\right| < \frac{M\left(K_{n+1}\right)}{\delta_n}\delta = \frac{\varepsilon M\left(K_{n+1}\right)}{\varepsilon + M\left(K_{n+1}\right)} < \varepsilon.$$
- Therefore we have
$$\left|f\left(z'\right) - f\left(z''\right)\right| < \varepsilon \tag{*}$$
for $f \in \mathcal{F}$ and $z', z'' \in K_n$ and $|z' - z''| < \delta$.
- Thus the family $\mathcal{F}$ is equicontinuous on compact subsets of $\Omega$, since every compact subset $K$ of $\Omega$ is contained in $K_n$ for some $n \in \mathbb{N}$, and therefore (*) holds for all $f \in \mathcal{F}$ and $z', z'' \in K$ so that $|z' - z''| < \delta$.

# Montel lemma

- Let $(f_m)_{m\in\mathbb{N}} \subseteq \mathcal{F}$. We show that it has a subsequence which converges uniformly on compact subsets of $\Omega$.
- Let $E$ be a countable dense set of $\Omega$.
- For example, we take $E$ to be the set of all points of $\Omega$ with rational coordinates. We arrange the elements of $E$ as $w_1, w_2, w_3, w_4, \ldots$.
- Since $(f_m(w_1))_{m\in\mathbb{N}}$ is a bounded sequence by assumption on $\mathcal{F}$, the sequence $(f_m)_{m\in\mathbb{N}}$ has a subsequence $(f_{m1})_{m\in\mathbb{N}}$ so that $(f_{m1}(w_1))_{m\in\mathbb{N}}$ converges. Here, we have used the Bolzano–Weierstrass theorem.
- Similarly, the sequence $(f_{m1})_{m\in\mathbb{N}}$ has a subsequence $(f_{m2})_{m\in\mathbb{N}}$ such that $(f_{m2}(w_2))_{m\in\mathbb{N}}$ converges.
- Proceeding recursively, we see that for $i \in \mathbb{N}$ there is a subsequence $(f_{mi})_{i\in\mathbb{N}}$ of $(f_{m,i-1})_{i\in\mathbb{N}}$ such that $(f_{mi}(w_i))_{i\in\mathbb{N}}$ converges.
- Here we write $f_{m0}$ for $f_m$.
- Now the diagonal sequence $(f_{mm})_{m\in\mathbb{N}}$ converges at all $w \in E$.
- We show that $(f_{mm})_{m\in\mathbb{N}}$ converges uniformly on $K_n$ for any $n \in \mathbb{N}$.

# Montel lemma

- Then the diagonal sequence $(f_{mm})_{m\in\mathbb{N}}$ converges uniformly on all compact subsets $K$ of $\Omega$ since $K \subseteq K_n$ for some $n \in \mathbb{N}$.
- For $n \in \mathbb{N}$ and $\delta = \delta(n)$ as above, we have

$$K_n \subseteq \bigcup_{z\in K_n} D(z,\delta).$$

- Then

$$K_n \subseteq \bigcup_{z\in E\cap K_n} D(z,\delta),$$

  since $E \cap K_n$ is dense in $K_n$. Since $K_n$ is compact, we observe that the above open cover admits a finite subcover.
- Thus there exist $z_1, z_2, \ldots, z_p \in E \cap K_n$ such that

$$K_n \subseteq D\left(z_1,\delta\right) \cup D\left(z_2,\delta\right) \cup \cdots \cup D\left(z_p,\delta\right).$$

# Montel lemma

- For $\varepsilon > 0$, there exists $M$ depending only on $\varepsilon$ and $K_n$ such that
$$\left|f_{rr}\left(z_i\right) - f_{ss}\left(z_i\right)\right| \leq \varepsilon$$
for $r \geq M, s \geq M$ and $1 \leq i \leq p$.
- Let $z \in K_n$. Then $z \in D\left(z_i, \delta\right)$ for some $i$ with $1 \leq i \leq p$.
- Further, by the equicontinuity, we have
$$\begin{aligned} &|f_{rr}(z) - f_{ss}(z)| \\ &\qquad \leq |f_{rr}(z) - f_{rr}\left(z_i\right)| + |f_{rr}\left(z_i\right) - f_{ss}\left(z_i\right)| + |f_{ss}(z) - f_{ss}\left(z_i\right)| \\ &\qquad < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \end{aligned}$$
wherever $r \geq M, s \geq M$. Thus $(f_{mm})_{m\in\mathbb{N}}$ converges uniformly on $K_n$.
- Since every compact subset of $\Omega$ is contained in $K_n$ for some $n \in \mathbb{N}$, we conclude that $(f_{mm})_{m\in\mathbb{N}}$ converges uniformly on compact subsets of $\Omega$. This completes the proof of Montels' lemma. □

# Proof of the Riemann mapping theorem

**Proof:** Assume that $\Omega \neq \mathbb{C}$ is a simply connected region. Let $z_0 \in \Omega$ and define

$$\Sigma = \{\psi \in H(\Omega) : \psi \text{ is one-to-one on } \Omega \text{ and } \psi(\Omega) \subseteq D\}.$$

Our aim is to prove that $\Sigma$ contains an element which is onto.

- In fact, we will show that for $\psi \in \Sigma$, which is not onto $D$, there exists $\psi_1 \in \Sigma$ such that
$$\left|\psi_1'(z_0)\right| > \left|\psi'(z_0)\right|. \qquad (*)$$
- Next we consider
$$\eta = \sup\left\{\left|\psi'(z_0)\right| : \psi \in \Sigma\right\}, \qquad (**)$$
and we will prove that the supremum is assumed for some $\psi_0 \in \Sigma$.
- Then it will be clear that $\psi_0$ has to be an onto function. Otherwise, if $\psi_0$ is not onto $D$ then there is $\psi_1 \in \Sigma$ satisfying (*). Hence, by (**), we have $\eta = |\psi_0'(z_0)| < |\psi_1'(z_0)| \leq \eta$, which is impossible.

The proof will consist of a few steps.

# Proof of the Riemann mapping theorem

**Step 1.** We first prove that $\Sigma \neq \emptyset$.

- From Lecture 8 we know that if $\Omega \subseteq \mathbb{C}$ is a simply connected region, then for every closed curve $\Gamma$ in $\Omega$ we have

$$\mathrm{Ind}_\Gamma(\alpha) = 0 \quad \text{whenever} \quad \alpha \notin \Omega.$$

- From Lecture 9, we know that the latter is equivalent to the statement that for every $f \in H(\Omega)$ satisfying $1/f \in H(\Omega)$ there exists $g \in H(\Omega)$ such that $f = g^2$.
- Since $\Omega \neq \mathbb{C}$, let $w_0 \in \mathbb{C}$ such that $w_0 \notin \Omega$. Consider

$$f(z) = z - w_0.$$

- We observe that $f \in H(\Omega)$ and $f$ has no zero in $\Omega$ since $w_0 \notin \Omega$.
- Thus $1/f \in H(\Omega)$. Therefore, there exists $g \in H(\Omega)$ such that

$$f(z) = g^2(z) \quad \text{for} \quad z \in \Omega.$$

# Proof of the Riemann mapping theorem

- Let $a \in \Omega$. By open mapping theorem, we observe that $g(\Omega)$ is open containing $g(a)$.
- Therefore there exists $r > 0$ such that

$$D(g(a), r) \subseteq g(\Omega).$$

- Now we show that $D(-g(a), r) \cap g(\Omega) = \emptyset$. Suppose there exists $z_1 \in \Omega$ such that $g(z_1) \in D(-g(a), r)$. Thus

$$|g(z_1) + g(a)| < r \quad \Longleftrightarrow \quad |-g(z_1) - g(a)| < r$$

- Then

$$-g(z_1) \in D(g(a), r) \subseteq g(\Omega).$$

- Therefore there exists $z_2 \in \Omega$ such that $-g(z_1) = g(z_2)$.
- By squaring both sides of this equation, we derive that

$$z_1 - w_0 = f(z_1) = (-g(z_1))^2 = (g(z_2))^2 = f(z_2) = z_2 - w_0$$

implying $z_1 = z_2$. Thus $g(z_1) = 0$, and consequently $z_1 - w_0 = 0$.

# Proof of the Riemann mapping theorem

- This is a contradiction since $z_1 \in \Omega$ and $w_0 \notin \Omega$. Hence,

$$D(-g(a), r) \cap g(\Omega) = \emptyset.$$

- Now we consider

$$\psi(z) = \frac{r}{g(z) + g(a)} \quad \text{for} \quad z \in \Omega.$$

- We observe that $\psi(z) \in H(\Omega)$ and $|\psi(z)| \leq 1$ for $z \in \Omega$, since

$$|g(z) + g(a)| \geq r \quad \text{for} \quad z \in \Omega.$$

- In fact $|\psi(z)| < 1$ for $z \in \Omega$ by the maximum modulus principle.
- Further $\psi \in H(\Omega)$ and is one-to-one on $\Omega$, since $\psi\left(z'\right) = \psi\left(z''\right)$ implies $g^2\left(z'\right) = g^2\left(z''\right)$, and therefore $z' = z''$.
- Hence $\psi \in \Sigma$ as desired.

# Proof of the Riemann mapping theorem

**Step 2.** We will show that if $\psi \in \Sigma$ and $\psi(\Omega)$ is a proper subset of $D$, then there exists $\psi_1 \in \Sigma$ satisfying

$$\left|\psi_1'(z_0)\right| > \left|\psi'(z_0)\right|. \tag{*}$$

- Fix $\psi \in \Sigma$. Since $\psi(\Omega)$ is a proper subset of $D$, there exists $\alpha \in D$ such that $\alpha \notin \psi(\Omega)$. We consider $\phi_\alpha \circ \psi$, where

$$\phi_\alpha(w) = \frac{w - \alpha}{1 - \overline{\alpha} w}.$$

- We recall that $\phi_\alpha$ is an automorphism of $D$. For $z \in \Omega$, observe that

$$\phi_\alpha \circ \psi(z) = \phi_\alpha(\psi(z)) = \frac{\psi(z) - \alpha}{1 - \overline{\alpha}\psi(z)} = 0$$

only when $\psi(z) = \alpha$ which is not the case since $\alpha \notin \psi(\Omega)$.

# Proof of the Riemann mapping theorem

- Therefore $\phi_\alpha \circ \psi \in H(\Omega)$ has no zero in $\Omega$. Then, there exists $g \in H(\Omega)$ such that

$$g^2(z) = \phi_\alpha \circ \psi(z) \quad \text{for} \quad z \in \Omega.$$

- By writing $s(w) = w^2$ for $w \in D$, we rewrite the last equation as

$$s \circ g = \phi_\alpha \circ \psi \quad \text{in} \quad \Omega.$$

- If $g(z_1) = g(z_2)$ for $z_1, z_2 \in \Omega$, then $\psi(z_1) = \psi(z_2)$, since $\phi_\alpha$ is one-to-one. Therefore, $g$ is one-to-one, and $g \in \Sigma$.
- Let

$$\psi_1 = \phi_\beta \circ g \quad \text{with} \quad g(z_0) = \beta.$$

- Observe that

$$\psi_1(z_0) = \phi_\beta(g(z_0)) = \phi_\beta(\beta) = 0.$$

# Proof of the Riemann mapping theorem

- Hence,

$$\psi = \phi_{-\alpha} \circ s \circ g = \phi_{-\alpha} \circ s \circ \phi_{-\beta} \circ \psi_1 = F \circ \psi_1,$$

where

$$F = \phi_{-\alpha} \circ s \circ \phi_{-\beta}.$$

- By the chain rule and $\psi_1(z_0) = 0$, we have

$$\psi'(z_0) = F'(\psi_1(z_0))\,\psi_1'(z_0) = F'(0)\psi_1'(z_0).$$

- Inequality (*) will follow if we show that $|F'(0)| < 1$, since

$$\left|\psi'(z_0)\right| = \left|F'(0)\right| \left|\psi_1'(z_0)\right|.$$

- Recall the following lemma from the previous lecture.

# Proof of the Riemann mapping theorem

**Lemma**

*Let $f$ be non-constant and analytic in $D$, and satisfy $|f(z)| < 1$ for $z \in D$. Let $w \in D$ with $f(w) = a$. Then*

$$\left|f'(w)\right| \leq \frac{1 - |a|^2}{1 - |w|^2}.$$

*Moreover equality occurs only when*

$$f = \phi_{-a} \circ (c\phi_w) \text{ in } D,$$

*for some constant $c$ whose absolute value is 1.*

- We observe that $F(D) \subseteq D$, and let $F(0) = a$. Then by the lemma with $w = 0$, we have
$$\left|F'(0)\right| \leq 1 - |a|^2.$$

# Proof of the Riemann mapping theorem

- Suppose that $|F'(0)| = 1$. Then $a = 0$ and by the second part of the previous lemma, we conclude with $w = 0$ that $F(z) = \lambda z$ for $z \in D$ where $\lambda$ is a constant of absolute value 1.
- This is not possible since $F$ is not one-to-one as $s(w) = w^2$ is not one-to-one. Hence $|F'(0)| < 1$ and the proof of (*) is complete.

**Step 3.** In this step we finish the proof. In Step 1, we have proved that $\Sigma \neq \emptyset$, hence we can define

$$\eta = \sup \left\{ \left|\psi'\left(z_0\right)\right| : \psi \in \Sigma \right\}, \qquad (**)$$

- By the inverse mapping theorem we have $|\psi'\left(z_0\right)| > 0$ for every $\psi \in \Sigma$. Hence $\eta > 0$.
- There exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq \Sigma$ such that

$$\lim_{n\to\infty} \left|\psi_n'\left(z_0\right)\right| = \eta > 0.$$

# Proof of the Riemann mapping theorem

- We observe that $|\psi(z)| < 1$ for $\psi \in \Sigma$.
- In particular, $\Sigma$ is uniformly bounded on compact subsets of $\Omega$.
- Therefore $\Sigma$ is a normal family by Montel's lemma.
- Hence the above sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq \Sigma$ has a subsequence which we denote again by $(\psi_n)_{n\in\mathbb{N}} \subseteq \Sigma$, and which converges uniformly on compact subsets of $\Omega$ satisfying

$$\lim_{n\to\infty} \left|\psi_n'(z_0)\right| = \eta > 0.$$

- Let

$$\lim_{n\to\infty} \psi_n(z) = h(z) \quad \text{for} \quad z \in \Omega$$

  converge uniformly on compact subsets of $\Omega$. Then $h \in H(\Omega)$.
- Further

$$\lim_{n\to\infty} \psi_n'(z) = h'(z) \quad \text{for} \quad z \in \Omega$$

  converges uniformly on compact subsets $\Omega$.

# Proof of the Riemann mapping theorem

- Therefore
$$\lim_{n\to\infty} \left|\psi_n'(z_0)\right| = \left|h'(z_0)\right|,$$
which implies
$$\left|h'(z_0)\right| = \eta > 0.$$
- Hence $h$ cannot be constant, otherwise we would have $\eta = 0$.
- Therefore $|h(z)| < 1$ for $z \in \Omega$ by the maximum modulus principle, in other words
$$h(\Omega) \subseteq D.$$
- By the Hurwitz lemma $h$ must be also one-to-one, thus $h \in \Sigma$ and is the maximizer for (**) as desired.
- Hence $h(\Omega) = D$ as it was explained by combining (*) and (**) at the beginning of the proof.
- This completes the proof of the Riemann mapping theorem. □