Lecture 15

The Runge theorem

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The Riemann mapping theorem

- $D = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc centered at the origin.

Theorem

Let $\Omega \neq \mathbb{C}$ be a simply connected region. Then Ω is conformally equivalent to D. Moreover, the assumption $\Omega \neq \mathbb{C}$ is necessary.

Remark

- In view of the Liouville theorem, the assumption $\Omega \neq \mathbb{C}$ is necessary. Indeed, if $f : \mathbb{C} \to D$ is a conformal map, then f is bounded, since $|f(z)| < 1$ for all $z \in \mathbb{C}$. Hence, by the Liouville theorem f must be constant, but then it cannot be injective.
- However, $\mathbb{C}$ and D are homeomorphic. The mapping

$$\mathbb{C} \ni z \mapsto \frac{z}{1 + |z|} \in D$$

is the desired homeomorphism.

Rational functions

Definition

- A **rational function** f is, by definition, a quotient of two polynomials P and Q, i.e.

$$f = P/Q.$$

- By the fundamental theorem of algebra every nonconstant polynomial is a product of factors of degree 1.
- Thus, we may assume that P and Q have no such factors in common.
- Then f has a pole at each zero of Q (the pole of f has the same order as the zero of Q).
- If we subtract the corresponding principal parts, we obtain a rational function whose only singularity is at ∞ and which is therefore a polynomial.

Rational functions

Definition

- Every **rational function**
$$f = P/Q,$$
has thus a representation of the form
$$f(z) = A_0(z) + \sum_{j=1}^{k} A_j\left((z-a_j)^{-1}\right), \qquad (*)$$
where $A_0, A_1, \ldots, A_k$ are polynomials, $A_1, \ldots, A_k$ have no constant term, and $a_1, \ldots, a_k$ are the distinct zeros of Q;
- Representation (*) is called the **partial fractions decomposition** of the rational function f.

Sets of oriented intervals

- Let Φ be a finite collection of oriented intervals in the plane.
- For each point p, let $m_I(p)$, [resp. $m_E(p)$] be the number of members of Φ that have initial point [resp. end point] p. If $m_I(p) = m_E(p)$ for every p, we shall say that Φ is **balanced**.

Construction

If $\Phi \neq \emptyset$ is balanced, the following construction can be carried out.

- Pick $\gamma_1 = [a_0, a_1] \in \Phi$. Assume $k \geq 1$, and assume that distinct members $\gamma_1, \ldots, \gamma_k$ of Φ have been chosen in such a way that $\gamma_i = [a_{i-1}, a_i]$ for $1 \leq i \leq k$. If $a_k = a_0$, stop. If $a_k \neq a_0$, and if precisely r of the intervals $\gamma_1, \ldots, \gamma_k$ have a_k as end point, then only $r - 1$ of them have a_k as initial point; since Φ is balanced, Φ contains at least one other interval, say γ_{k+1}, whose initial point is a_k.
- Since Φ is finite, we must return to a_0 eventually, say at the nth step.
- Then join the intervals $\gamma_1, \ldots, \gamma_n$ (in this order) to form a closed path.

Sets of oriented intervals

- The remaining members of Φ still form a balanced collection to which the above construction can be applied.
- It follows that the members of Φ can be so numbered that they form finitely many closed paths. The sum of these paths is a cycle.
- Hence, we conclude, if $\Phi = \{\gamma_1, \ldots, \gamma_N\}$ is a balanced collection of oriented intervals, and if $\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_N$, then Γ is a cycle.

Theorem

If $K \subset \mathbb{C}$ is a compact subset of an open set $\Omega \neq \emptyset$, then there is a cycle Γ in $\Omega \setminus K$ such that the Cauchy formula

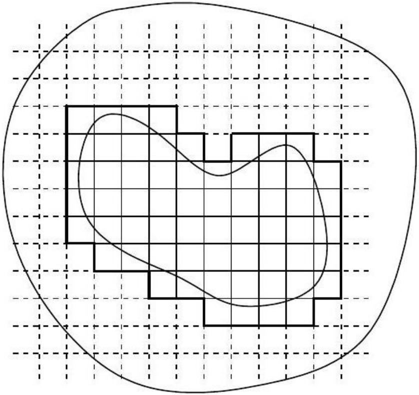
$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta \qquad (**)$$

holds for every $f \in H(\Omega)$ and for every $z \in K$.

Cauchy theorem for compact sets

Proof: Since K is compact and Ω is open, there exists an $\eta > 0$ such that the distance from any point of K to any point outside Ω is at least 2η.

- Construct a grid of horizontal and vertical lines in the plane, such that the distance between any two adjacent horizontal lines is η, and likewise for the vertical lines.



Cauchy theorem for compact sets

- Let $Q_1, \dots, Q_m$ be those squares (closed 2-cells) of edge η which are formed by this grid and which intersect K.
- Then $Q_r \subset \Omega$ for $r = 1, \dots, m$.
- If a_r is the center of Q_r and $a_r + b$ is one of its vertices, let γ_{rk} be the oriented interval

$$\gamma_{rk} = \left[a_r + i^k b, a_r + i^{k+1} b\right],$$

and define

$$\partial Q_r = \gamma_{r1} \dotplus \gamma_{r2} \dotplus \gamma_{r3} \dotplus \gamma_{r4} \quad \text{for} \quad r = 1, \dots, m.$$

Cauchy theorem for compact sets

- It is then easy to check that

$$\mathrm{Ind}_{\partial Q_r}(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is in the interior of } Q_r, \\ 0 & \text{if } \alpha \text{ is not in } Q_r. \end{cases}$$

- Let Σ be the collection of all γ_{rk} with $1 \leq r \leq m$, and $1 \leq k \leq 4$. It is clear that Σ is balanced.
- Remove those members of Σ whose opposites also belong to Σ.
- Let Φ be the collection of the remaining members of Σ. Then Φ is balanced. Let Γ be the cycle constructed from Φ.
- If an edge E of some Q_r intersects K, then the two squares in whose boundaries E lies intersect K. Hence Σ contains two oriented intervals which are each other's opposites and whose range is E. These intervals do not occur in Φ.
- Thus Γ is a cycle in $\Omega \setminus K$.

Cauchy theorem for compact sets

- The construction of Φ from Σ shows also that

$$\mathrm{Ind}_{\Gamma}(\alpha) = \sum_{r=1}^{m} \mathrm{Ind}_{\partial Q_r}(\alpha),$$

if α is not in the boundary of any Q_r. Hence, we obtain

$$\mathrm{Ind}_{\Gamma}(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is in the interior of some } Q_r, \\ 0 & \text{if } \alpha \text{ lies in no } Q_r. \end{cases}$$

- If $z \in K$, then $z \notin \Gamma^*$, and z is a limit point of the interior of some Q_r. Since the left side of $\mathrm{Ind}_{\Gamma}(\alpha)$ is constant in each component of the complement of Γ^*, we obtain

$$\mathrm{Ind}_{\Gamma}(z) = \begin{cases} 1 & \text{if } z \in K, \\ 0 & \text{if } z \notin \Omega. \end{cases}$$

- Now (**) follows from the global Cauchy theorem. □

Runge's theorem

Theorem

Suppose $K \subset \mathbb{C}$ *is a compact set and* $(\alpha_j)_{j\in\mathbb{N}}$ *is a set which contains one point in each component of* $\mathbb{C}_\infty \setminus K$. *If* Ω *is open, and* $\Omega \supset K$, *and* $f \in H(\Omega)$, *and* $\varepsilon > 0$, *there exists a rational function* R, *all of whose poles lie in the prescribed set* $(\alpha_j)_{j\in\mathbb{N}}$, *such that*

$$|f(z) - R(z)| < \varepsilon$$

for every $z \in K$.

Remark

- Note that $\mathbb{C}_\infty \setminus K$ has at most countably many components.
- Note also that the preassigned point in the unbounded component of $\mathbb{C}_\infty \setminus K$ may very well be ∞; in fact, this happens to be the most interesting choice.

Runge's theorem

Proof: We consider the Banach space $C(K)$ whose members are the continuous complex functions on K, with the supremum norm. Let M be the linear subspace of $C(K)$ which consists of the restrictions to K of those rational functions which have all their poles in $(\alpha_j)_{j\in\mathbb{N}}$.

- The theorem asserts that f is in the closure of M.
- As a consequence of the Hahn–Banach theorem, this is equivalent to saying that every bounded linear functional Λ on $C(K)$ which vanishes on M also vanishes at f, i.e.

$$\Lambda(f) = 0.$$

- By the the Riesz representation theorem (it is covered in 502) there exists a complex Borel measure μ on K such

$$\Lambda(f) = \int_K f(x) d\mu(x) \quad \text{for every} \quad f \in C(K).$$

Runge's theorem

Therefore we have to prove the following assertion:

- If μ is a complex Borel measure on K such that

$$\int_K R d\mu = 0 \quad \text{for every} \quad R \in M,$$

  and if $f \in H(\Omega)$, then we also have

$$\int_K f d\mu = 0.$$

- So let us assume that μ is a complex Borel measure on K as above, and define

$$h(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z} \quad \text{for} \quad z \in \mathbb{C}_\infty \setminus K.$$

- Then $h \in H(\mathbb{C}_\infty \setminus K)$. (Why?)

Runge's theorem

- Let V_j be the component of $\mathbb{C}_\infty \setminus K$ which contains α_j, and suppose $D(\alpha_j; r) \subset V_j$. If $\alpha_j \neq \infty$ and if z is fixed in $D(\alpha_j; r)$, then

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - \alpha_j} \frac{1}{1 - \frac{z-\alpha_j}{\zeta-\alpha_j}} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(z-\alpha_j)^n}{(\zeta-\alpha_j)^{n+1}}$$

  uniformly for $\zeta \in K$.

- Now observe that

$$h(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z} = \lim_{N\to\infty} \sum_{n=0}^{N} \int_K \frac{(z-\alpha_j)^n}{(\zeta-\alpha_j)^{n+1}} d\mu(\zeta) = 0,$$

  since $(z-\alpha_j)^n (\zeta-\alpha_j)^{-n-1} \in M$, and $\int_K R d\mu = 0$ for every $R \in M$.

- Hence $h(z) = 0$ for all $z \in D(\alpha_j; r)$. This implies that $h(z) = 0$ for all $z \in V_j$, by the uniqueness theorem.

Runge's theorem

- If $\alpha_j = \infty$, and if z is fixed in $D(\infty; r)$, i.e. $|z| > r$, then

$$\frac{1}{\zeta - z} = -\frac{1}{z}\frac{1}{1 - \frac{\zeta}{z}} = -\lim_{N\to\infty}\sum_{n=0}^{N} z^{-n-1}\zeta^n$$

uniformly for $\zeta \in K$.

- Hence, as before, we deduce that $h(z) = 0$ in $D(\infty; r)$, hence in V_j.
- We have thus proved that

$$h(z) = 0 \quad \text{for} \quad z \in \mathbb{C}_\infty \setminus K.$$

- Now choose a cycle Γ in $\Omega \setminus K$, as in the previous theorem, then

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

holds for every $z \in K$.

Runge's theorem

- An application of Fubini's theorem (legitimate, since we are dealing with Borel measures and continuous functions on compact spaces), combined with $h(z) = 0$ for $z \in \mathbb{C}_\infty \setminus K$, gives

$$\begin{aligned}
\int_K f(\zeta) d\mu(\zeta) &= \int_K \left[\frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - \zeta} dw \right] d\mu(\zeta). \\
&= \frac{1}{2\pi i} \int_\Gamma f(w) dw \int_K \frac{d\mu(\zeta)}{w - \zeta} \\
&= -\frac{1}{2\pi i} \int_\Gamma f(w) h(w) dw = 0,
\end{aligned}$$

  since $h(w) = 0$ on $\Gamma^* \subset \Omega \setminus K \subseteq \mathbb{C}_\infty \setminus K$.

- Thus

$$\int_K f(\zeta) d\mu(\zeta)$$

  holds for any $f \in H(\Omega)$, and the proof is complete. □

Runge's theorem (approximation by polynomials)

The following special case is of particular interest.

Theorem

Suppose $K \subset \mathbb{C}$ is a compact set, $\mathbb{C}_\infty \setminus K$ is connected, and $f \in H(\Omega)$, where Ω is some open set containing K. Then there is a sequence $(P_n)_{n\in\mathbb{N}}$ of polynomials such that $\lim_{n\to\infty} P_n(z) = f(z)$ uniformly on K.

Proof: Since $\mathbb{C}_\infty \setminus K$ has now only one component, we need only one point α_j to apply the previous theorem, and we may take $\alpha_j = \infty$.

Runge's theorem

Remark

- The preceding result is **false** for every compact K in the plane such that $\mathbb{C}_\infty \setminus K$ is not connected.
- Namely, in that case $\mathbb{C}_\infty \setminus K$ has a bounded component V. Choose $\alpha \in V$, set $f(z) = (z-\alpha)^{-1}$, and let $m = \max\{|z-\alpha| : z \in K\}$.
- Suppose that P is a polynomial, such that $|P(z) - f(z)| < 1/m$ for all $z \in K$. Then

$$|(z-\alpha)P(z) - 1| < 1 \quad \text{for} \quad z \in K \tag{A}$$

- In particular, (A) holds if z is in the boundary of V; since the closure of V is compact, the maximum modulus theorem shows that (A) holds for every $z \in V$; taking $z = \alpha$, we obtain $1 < 1$.
- Hence the uniform approximation is not possible.

Runge's theorem

Recall the following lemma from the previous lecture:

Lemma

Let Ω be an open subset of $\mathbb{C}$. Then there exists a sequence $(K_n)_{n\in\mathbb{N}} \subseteq \Omega$ of compact sets such that

$$\Omega = \bigcup_{n=1}^{\infty} K_n \quad \textit{and} \quad K_n \subseteq \operatorname{int} K_{n+1} \quad \textit{for} \quad n \in \mathbb{N},$$

where $\operatorname{int} K_{n+1}$ *denotes the interior of K_{n+1}. Further for a compact set $K \subseteq \Omega$, we have $K \subseteq K_n$ for some $n \in \mathbb{N}$.*

Remark

From this lemma we can also deduce that every component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus \Omega$ for $n \in \mathbb{N}$.

Runge's theorem

- This can be seen by recalling that for every $n \in \mathbb{N}$, we defined

$$K_n = \overline{D}(0, n) \cap \{z \in \Omega : d(z, \mathbb{C} \setminus \Omega) \geq 1/n\}.$$

- Setting $D(\infty, n) = \{z \in \mathbb{C} : |z| > n\}$, we see

$$V_n = K_n^c = D(\infty, n) \cup \bigcup_{a \notin \Omega} D(a, 1/n).$$

- Therefore, each disc from V_n intersects $\mathbb{C}_\infty \setminus \Omega$.
- Each disc from V_n is connected, hence each component of V_n intersects $\mathbb{C}_\infty \setminus \Omega$.
- Since $V_n \supseteq \mathbb{C}_\infty \setminus \Omega$, no component of $\mathbb{C}_\infty \setminus \Omega$ can intersect two components of V_n.
- Hence, every component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus \Omega$ for $n \in \mathbb{N}$.

Runge's theorem

Theorem

Let $\Omega \subseteq \mathbb{C}$ be an open set, let A be a set which has one point in each component of $\mathbb{C}_\infty \setminus \Omega$, and assume that $f \in H(\Omega)$. Then:

- *There is a sequence $(R_n)_{n\in\mathbb{N}}$ of rational functions, with poles only in A, such that $\lim_{n\to\infty} R_n = f$ uniformly on compact subsets of Ω.*
- *In the special case in which $\mathbb{C}_\infty \setminus \Omega$ is connected, we may take $A = \{\infty\}$ and thus obtain polynomials P_n such that $\lim_{n\to\infty} P_n = f$ uniformly on compact subsets of Ω.*

Remark

- Observe that $\mathbb{C}_\infty \setminus \Omega$ may have uncountably many components; for instance, we may have $\mathbb{C}_\infty \setminus \Omega = \{\infty\} \cup C$, where C is a Cantor set.

Runge's theorem

Proof: Choose a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ in Ω, as in the previous lemma. Fix $n \in \mathbb{N}$, for the moment.

- Since each component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus \Omega$, each component of $\mathbb{C}_\infty \setminus K_n$ contains a point of A, so the previous theorem gives us a rational function R_n with poles in A such that

$$|R_n(z) - f(z)| < \frac{1}{n} \quad \text{for all} \quad z \in K_n.$$

- If now $K \subset \Omega$ is compact, there exists an $N \in \mathbb{N}$ such that $K \subset K_n$ for all $n \geq N$. It follows from the previous inequality that

$$|R_n(z) - f(z)| < \frac{1}{n} \quad \text{for all} \quad z \in K, \text{ and } n \geq N,$$

which completes the proof. □

Mittag–Leffler theorem

Runge's theorem will now be used to prove that meromorphic functions can be constructed with arbitrarily preassigned poles.

Theorem

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, $A \subset \Omega$, and A has no limit point in Ω, and to each $\alpha \in A$ there are associated a positive integer $m(\alpha)$ and a rational function

$$P_\alpha(z) = \sum_{j=1}^{m(\alpha)} c_{j,\alpha}(z-\alpha)^{-j}.$$

Then there exists a meromorphic function f in Ω, whose principal part at each $\alpha \in A$ is P_α and which has no other poles in Ω.

Proof: We choose a sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets in Ω, as in the previous lemma. Then $K_n \subseteq \operatorname{int} K_{n+1}$ for all $n \in \mathbb{N}$, and every compact subset of Ω lies in some K_n, for some $n \in \mathbb{N}$. Moreover, every component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus \Omega$.

Mittag–Leffler theorem

- Set $A_1 = A \cap K_1$, and $A_n = A \cap (K_n \setminus K_{n-1})$ for $n \geq 2$.
- Since $A_n \subset K_n$ and A has no limit point in Ω (hence none in K_n), each A_n is a finite set.
- Set

$$Q_n(z) = \sum_{\alpha \in A_n} P_\alpha(z) \quad \text{for} \quad n \in \mathbb{N}.$$

- Since each A_n is finite, each Q_n is a rational function. The poles of Q_n lie in $K_n \setminus K_{n-1}$, for $n \geq 2$. In particular, Q_n is holomorphic in an open set containing K_{n-1}.
- It now follows from Runge's theorem that there exist rational functions R_n, all of whose poles are in $\mathbb{C}_\infty \setminus \Omega$, such that

$$|R_n(z) - Q_n(z)| < 2^{-n} \quad \text{for all} \quad z \in K_{n-1}.$$

Mittag–Leffler theorem

- We claim that
$$f(z) = Q_1(z) + \sum_{n=2}^{\infty} (Q_n(z) - R_n(z)) \quad \text{for all} \quad z \in \Omega$$
has the desired properties.
- Fix $N \in \mathbb{N}$. On K_N, we have
$$f = Q_1 + \sum_{n=2}^{N} (Q_n - R_n) + \sum_{N+1}^{\infty} (Q_n - R_n) \, .$$
- Since $|R_n(z) - Q_n(z)| < 2^{-n}$ on K_{n-1}, then each term in the last sum is less than 2^{-n} on K_N; hence this last series converges uniformly on K_N, to a function which is holomorphic in the interior of K_N.
- The function $f - (Q_1 + \cdots + Q_N)$ is holomorphic in the interior of K_N, since the poles of each R_n are outside Ω.
- Thus f has precisely the prescribed principal parts in the interior of K_N, and hence in Ω, since N was arbitrary. □

Characterization of simply connectedness

Theorem

For a region $\Omega \subseteq \mathbb{C}$, each of the following conditions implies all the others.

(a) *Ω is homeomorphic to the open unit disc D.*

(b) *Ω is simply connected.*

(c) $\mathrm{Ind}_\gamma(\alpha) = 0$ *for every closed path γ in Ω and for every $\alpha \in \mathbb{C}_\infty \setminus \Omega$.*

(d) *$\mathbb{C}_\infty \setminus \Omega$ is connected.*

(e) *Every $f \in H(\Omega)$ can be approximated by polynomials, uniformly on compact subsets of Ω.*

(f) *For every $f \in H(\Omega)$ and every closed path γ in Ω we have $\int_\gamma f(z)dz = 0$.*

(g) *To every $f \in H(\Omega)$ corresponds an $F \in H(\Omega)$ such that $F' = f$.*

(h) *If $f \in H(\Omega)$ and $1/f \in H(\Omega)$, there exists a $g \in H(\Omega)$ such that $f = \exp(g)$.*

(i) *If $f \in H(\Omega)$ and $1/f \in H(\Omega)$, there exists a $\varphi \in H(\Omega)$ such that $f = \varphi^2$.*

Characterization of simply connectedness

Proof (a) $\implies$ (b). To say that Ω is homeomorphic to D means that there is a continuous one-to-one mapping $\psi : \Omega \to D$ whose inverse ψ^{-1} is also continuous.

- If γ is a closed curve in Ω, with parameter interval $[0,1]$, define

$$H(s,t) = \psi^{-1}(t\psi(\gamma(s)))$$

  Then $H : I^2 \to \Omega$ is continuous; $H(s,0) = \psi^{-1}(0)$ is constant; and $H(s,1) = \gamma(s)$; and $H(0,t) = H(1,t)$ because $\gamma(0) = \gamma(1)$.
- Thus γ is null-homotopic in Ω, hence Ω is simply connected.

Proof (b) $\implies$ (c). If (b) holds then every closed curve Γ in Ω is null-homotopic, that is, there exists a continuous $H : I^2 \to \Omega$ such that

$$H(s,0) = \Gamma(s), \quad H(s,1) = \gamma(s), \quad \text{for all} \quad s \in I,$$
$$H(0,t) = H(1,t) \quad \text{for all} \quad t \in I,$$

where γ is a constant curve. Hence $\mathrm{Ind}_\gamma(\alpha) = 0$ whenever $\alpha \notin \Omega$.

Characterization of simply connectedness

- Recall the following result:

Theorem

If Γ_0 and Γ_1 are Ω-homotopic closed paths in a region $\Omega \subseteq \mathbb{C}$, and if $\alpha \notin \Omega$, then

$$\mathrm{Ind}_{\Gamma_1}(\alpha) = \mathrm{Ind}_{\Gamma_0}(\alpha).$$

- Invoking this result, we obtain that $\mathrm{Ind}_{\Gamma}(\alpha) = \mathrm{Ind}_{\gamma}(\alpha) = 0$ for $\alpha \notin \Omega$ as desired completing the proof of implication from (b) to (c). □

Proof (c) $\implies$ (d). Assume (d) is false. Then $\mathbb{C}_\infty \setminus \Omega$ is a closed subset of $\mathbb{C}_\infty$ which is not connected. It follows that $\mathbb{C}_\infty \setminus \Omega$ is the union of two nonempty disjoint closed sets H and K.

- Let H be the one that contains ∞. Let W be the complement of H, relative to the plane. Then $W = \Omega \cup K$. Since K is compact, Cauchy theorem for compact sets (with $f = 1$) shows that there is a cycle Γ in $W \setminus K = \Omega$ such that $\mathrm{Ind}_{\Gamma}(z) = 1$ for $z \in K$.
- Since $K \neq \emptyset$, (c) fails, and we are done. □

Characterization of simply connectedness

Proof (d) $\implies$ (e). This is part follows from Runge's theorem. □

Proof (e) $\implies$ (f). Choose $f \in H(\Omega)$, let γ be a closed path in Ω, and choose polynomials P_n which converge to f, uniformly on γ^*. Since $\int_\gamma P_n(z)dz = 0$ for all $n \in \mathbb{N}$, we conclude that (f) holds. □

Proof (f) $\implies$ (g). Assume (f) holds, fix $z_0 \in \Omega$, and put

$$F(z) = \int_{\Gamma(z)} f(\zeta)d\zeta \quad \text{for all} \quad z \in \Omega,$$

where $\Gamma(z)$ is any path in Ω from z_0 to z.

- This defines a function F in Ω. For if $\Gamma_1(z)$ is another path from z_0 to z in Ω, then Γ followed by the opposite of Γ_1 is a closed path in Ω, the integral of f over this closed path is 0, so F is well-defined, i.e. F is not affected if $\Gamma(z)$ is replaced by $\Gamma_1(z)$.
- We now verify that $F' = f$. Fix $a \in \Omega$, then there exists an $r > 0$ such that $D(a; r) \subseteq \Omega$. For $z \in D(a; r)$ we can compute $F(z)$ by integrating f over a path $\Gamma(a)$, followed by the interval $[a, z]$.

Characterization of simply connectedness

- Hence, for $z \in D'(a;r)$, we have

$$\frac{F(z)-F(a)}{z-a} = \frac{1}{z-a}\int_{[a,z]} f(\zeta)d\zeta,$$

and the continuity of f at a implies now that $F'(a) = f(a)$.
Proof (g) $\implies$ (h). If $f \in H(\Omega)$ and f has no zero in Ω, then $f'/f \in H(\Omega)$, and (g) implies that there exists a $g \in H(\Omega)$ so that $g' = f'/f$. We can add a constant to g, so that $\exp(g(z_0)) = f(z_0)$ for some $z_0 \in \Omega$. Our choice of g shows that $(fe^{-g})' = 0$ in Ω, hence fe^{-g} is constant (since Ω is connected), and $f = e^g$. □
Proof (h) $\implies$ (i). By (h) we have $f = e^g$. Set $\varphi = \exp\left(\frac{1}{2}g\right)$. □
Proof (i) $\implies$ (a). If Ω is the whole plane, then Ω is homeomorphic to D, take $z \mapsto z/(1+|z|)$. If Ω is a proper subregion of the plane which satisfies (i), then there actually exists a holomorphic homeomorphism of Ω onto D by the Riemann mapping theorem. □