Lecture 17

Infinite products

MATH 503, FALL 2025

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Jensen's formula

Theorem

Let Ω be an open set that contains the closure of a disc $D(0,R)$ and suppose that f is holomorphic in Ω, and $f(0)\neq 0$, f vanishes nowhere on the circle $C(0,R)$. If $z_1,\dots,z_N$ denote the zeros of f inside the disc (counted with multiplicities, i.e. each zero appears in the sequence as many times as its order), then

$$\log|f(0)| = \sum_{k=1}^{N}\log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|d\theta. \qquad (*)$$

The proof of the theorem consists of several steps.

- **Step 1.** First, we observe that if f_1 and f_2 are two functions satisfying the hypotheses and the conclusion of the theorem, then the product $f_1 f_2$ also satisfies the hypothesis of the theorem and formula (*).

Jensen's formula

- This is a simple consequence of the fact that $\log xy = \log x + \log y$ whenever $x, y \in \mathbb{R}_+$, and that the set of zeros of $f_1 f_2$ is the union of the sets of zeros of f_1 and f_2.
- **Step 2.** The function

$$g(z) = \frac{f(z)}{(z - z_1) \cdots (z - z_N)}$$

initially defined on $\Omega \setminus \{z_1, \ldots, z_N\}$, is bounded near each z_j.
- Therefore each z_j is a removable singularity, and hence we can write

$$f(z) = (z - z_1) \cdots (z - z_N) \, g(z),$$

where g is holomorphic in Ω and nowhere vanishing in the closure of $D(0, R)$. By Step 1, it suffices to prove Jensen's formula for functions like g that vanish nowhere, and for functions of the form $z - z_j$.

Jensen's formula

- **Step 3.** We first prove (*) for a function g that vanishes nowhere in the closure of $D(0, R)$. More precisely, we must establish the following identity:

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|d\theta. \tag{**}$$

- In a slightly larger disc, we can write $g(z) = e^{h(z)}$ where h is holomorphic in that disc. This is possible since discs are simply connected, and we can define $h = \log g$.
- Now

$$|g(z)| = |e^{h(z)}| = |e^{\mathrm{Re}(h(z))+i\,\mathrm{Im}(h(z))}| = e^{\mathrm{Re}(h(z))},$$

so that $\log|g(z)| = \mathrm{Re}(h(z))$. Then the mean value property for holomorphic functions (in our case with $h = \log g$) immediately implies the desired formula for its real part, which is precisely (**).

Jensen's formula

- **Step 4.** The last step is to prove the formula for functions of the form $f(z) = z - w$, where $w \in D(0, R)$. That is, we must show that

$$\log|w| = \log\left(\frac{|w|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|Re^{i\theta} - w|d\theta.$$

- Since $\log(|w|/R) = \log|w| - \log R$ and

$$\log|Re^{i\theta} - w| = \log R + \log|e^{i\theta} - w/R|,$$

it suffices to prove that

$$\int_0^{2\pi} \log|e^{i\theta} - a|d\theta = 0, \quad \text{whenever} \quad |a| < 1.$$

- This in turn is equivalent (after the change of variables $\theta \mapsto -\theta$) to

$$\int_0^{2\pi} \log|1 - ae^{i\theta}|d\theta = 0, \quad \text{whenever} \quad |a| < 1.$$

Jensen's formula

- To prove this, we use the function $F(z) = 1 - az$, which vanishes nowhere in the closure of the unit disc.
- As a consequence, there exists a holomorphic function G in a disc of radius greater than 1 such that

$$F(z) = e^{G(z)}.$$

- Then $|F| = e^{\text{Re}(G)}$, and therefore $\log |F| = \text{Re}(G)$. Since $F(0) = 1$ we have $\log |F(0)| = 0$, and an application of the mean value property to the real part of holomorphic function G, which is $\log |F(z)|$ concludes the proof of the theorem, giving

$$\int_0^{2\pi} \log |1 - ae^{i\theta}| d\theta = 0, \quad \text{whenever} \quad |a| < 1. \qquad \square$$

Jensen's formula

- From Jensen's formula we can derive an identity linking the growth of a holomorphic function with its number of zeros inside a disc.
- If f is a holomorphic function on the closure of a disc $D(0, R)$, we denote by $\mathfrak{n}_f(r)$ the number of zeros of f (counted with their multiplicities) inside the disc $D(0, r)$, with $0 < r < R$.
- A simple but useful observation is that $\mathfrak{n}_f(r)$ is a non-decreasing function of r.

Lemma

Under the assumptions of the previous theorem, we have

$$\int_0^R \mathfrak{n}_f(r) \frac{dr}{r} = \sum_{k=1}^{N} \log \left| \frac{R}{z_k} \right|.$$

Jensen's formula

Proof:

- First we have
$$\sum_{k=1}^{N} \log\left|\frac{R}{z_k}\right| = \sum_{k=1}^{N} \int_{|z_k|}^{R} \frac{dr}{r}.$$
- If we define the characteristic function
$$\eta_k(r) = \begin{cases} 1 & \text{if } r > |z_k|, \\ 0 & \text{if } r \le |z_k|, \end{cases}$$

  then
$$\sum_{k=1}^{N} \eta_k(r) = \mathfrak{n}_f(r).$$

Jensen's formula

- The lemma is proved using

$$\sum_{k=1}^{N}\int_{|z_k|}^{R}\frac{dr}{r}=\sum_{k=1}^{N}\int_{0}^{R}\eta_k(r)\frac{dr}{r}=\int_{0}^{R}\left(\sum_{k=1}^{N}\eta_k(r)\right)\frac{dr}{r}=\int_{0}^{R}\mathfrak{n}_f(r)\frac{dr}{r}.$$

- This completes the proof of the lemma. □

Corollary

As a corollary of Jensen's formula and the previous lemma, we obtain

$$\int_{0}^{R}\mathfrak{n}_f(r)\frac{dr}{r}=\frac{1}{2\pi}\int_{0}^{2\pi}\log|f(Re^{i\theta})|d\theta-\log|f(0)|.$$

Functions of finite order

- Let f be an entire function. If there exist $\rho \in \mathbb{R}_+$ and constants $A, B \in \mathbb{R}_+$ such that

$$|f(z)| \leq Ae^{B|z|^\rho} \quad \text{for all} \quad z \in \mathbb{C},$$

  then we say that f has an **order of growth** $\leq \rho$.
- We define the **order of growth** of f as

$$\rho_f = \inf \rho,$$

  where the infimum is taken over all $\rho > 0$ such that f has an order of growth $\leq \rho$.
- For example, the order of growth of the function e^{z^2} is 2.

Functions of finite order

Theorem

If f is an entire function that has an order of growth $\leq \rho$, then:

(i) $\mathfrak{n}_f(r) \leq Cr^{\rho}$ *for some* $C > 0$ *and all sufficiently large* r.

(ii) *If* $z_1, z_2, \ldots$ *denote the zeros of* f, *with* $z_k \neq 0$, *then for all* $s > \rho$ *we have*

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty.$$

Proof: It suffices to prove the estimate for $\mathfrak{n}_f(r)$ when $f(0) \neq 0$.

- Indeed, consider $F(z) = f(z)/z^l$, where l is the order of the zero of f at the origin. Then $\mathfrak{n}_f(r)$ and $\mathfrak{n}_F(r)$ differ only by a constant, and F also has an of order of growth $\leq \rho$.

Functions of finite order

- If $f(0) \neq 0$ we may use formula from the previous corollary, namely

$$\int_0^R \mathfrak{n}_f(x)\frac{dx}{x} = \frac{1}{2\pi}\int_0^{2\pi} \log|\, f(Re^{i\theta})|d\theta - \log|\, f(0)|.$$

- Choosing $R = 2r$, this formula implies

$$\int_r^{2r} \mathfrak{n}_f(x)\frac{dx}{x} \leq \frac{1}{2\pi}\int_0^{2\pi} \log|\, f(Re^{i\theta})|d\theta - \log|\, f(0)|.$$

- On the one hand, since $\mathfrak{n}_f(r)$ is increasing, we have

$$\int_r^{2r} \mathfrak{n}_f(x)\frac{dx}{x} \geq \mathfrak{n}_f(r)\int_r^{2r} \frac{dx}{x} = \mathfrak{n}_f(r)[\log 2r - \log r] = \mathfrak{n}_f(r)\log 2.$$

- On the other hand, the growth condition on f (for all large r) gives

$$\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta \leq \int_0^{2\pi} \log|Ae^{BR^{\rho}}|d\theta \leq C' r^{\rho}.$$

Functions of finite order

- Consequently, $\mathfrak{n}_f(r) \leq Cr^\rho$ for an appropriate $C > 0$ and all sufficiently large r.
- The following estimates prove the second part of the theorem:

$$\begin{aligned}\sum_{|z_k|\geq 1} |z_k|^{-s} &= \sum_{j=0}^{\infty} \left(\sum_{2^j \leq |z_k| < 2^{j+1}} |z_k|^{-s} \right) \\ &\leq \sum_{j=0}^{\infty} 2^{-js} \mathfrak{n}_f \left(2^{j+1}\right) \\ &\leq c \sum_{j=0}^{\infty} 2^{-js} 2^{(j+1)\rho} \leq c' \sum_{j=0}^{\infty} \left(2^{\rho - s}\right)^j < \infty\end{aligned}$$

- The last series converges because $s > \rho$.

This completes the proof of the theorem. □

Infinite products

- Given a sequence $(a_n)_{n\in\mathbb{Z}_+} \subseteq \mathbb{C}$, we say that the product

$$\prod_{n=1}^{\infty}(1+a_n)$$

converges if the limit

$$\lim_{N\to\infty}\prod_{n=1}^{N}(1+a_n)$$

of the partial products exists.

- A useful necessary condition that guarantees the existence of a product is contained in the following lemma.

Lemma

If $\sum_{n\in\mathbb{Z}_+}|a_n| < \infty$, then the product $\prod_{n=1}^{\infty}(1+a_n)$ converges. Moreover, the product converges to 0 if and only if one of its factors is 0.

Infinite products

- If $\sum_{n\in\mathbb{Z}_+} |a_n|$ converges, then for all large n we must have $|a_n| < 1/2$. We may assume, without loss of generality, that this inequality holds for all $n \in \mathbb{Z}_+$.
- Hence, we can define $\log(1 + a_n)$ by the usual power series, and this logarithm satisfies the property that

$$1 + z = e^{\log(1+z)},$$

whenever $|z| < 1$.
- Hence we may write the partial products as follows:

$$\prod_{n=1}^{N} (1 + a_n) = \prod_{n=1}^{N} e^{\log(1+a_n)} = e^{B_N}$$

where $B_N = \sum_{n=1}^{N} b_n$ with $b_n = \log(1 + a_n)$.

Infinite products

- By the power series expansion we see that

$$|\log(1+z)| \leq 2|z|,$$

  if $|z| < 1/2$. Hence $|b_n| \leq 2\,|a_n|$, so B_N converges as $N \to \infty$ to a complex number, say B.

- Since the exponential function is continuous, we conclude that

$$\lim_{N\to\infty} e^{B_N} = e^B,$$

  and the first part follows.

- Observe also that if $1 + a_n \neq 0$ for all $n \in \mathbb{Z}_+$, then the product converges to a non-zero limit since it is expressed as e^B. $\square$

Infinite products of holomorphic functions

Lemma

Suppose $(F_n)_{n\in\mathbb{Z}_+}$ *is a sequence of holomorphic functions on the open set* $\Omega \subseteq \mathbb{C}$. *If there exist constants* $c_n > 0$ *such that*

$$\sum_{n\in\mathbb{Z}_+} c_n < \infty \quad \text{and} \quad |F_n(z) - 1| \leq c_n \quad \text{for all} \quad z \in \Omega,$$

then:

(i) *The product* $\prod_{n=1}^{\infty} F_n(z)$ *converges uniformly in* Ω *to a holomorphic function* $F(z)$.

(ii) *If* $F_n(z)$ *does not vanish for any* n, *then*

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}.$$

Infinite products of holomorphic functions

- To prove the first statement, note that for each z we may argue as in the previous lemma if we write $F_n(z) = 1 + a_n(z)$, with $|a_n(z)| \leq c_n$.
- Then, we observe that the estimates are actually uniform in z because the c_n 's are constants. It follows that the product converges uniformly to a holomorphic function, which we denote by $F(z)$.
- To establish the second part of the theorem, suppose that K is a compact subset of Ω, and let

$$G_N(z) = \prod_{n=1}^{N} F_n(z).$$

- We have just proved that $\lim_{N\to\infty} G_N = F$ uniformly in Ω. Hence, the sequence $(G_N')_{N\in\mathbb{Z}_+}$ converges uniformly to F' in K.

Infinite products of holomorphic functions

- Since G_N is uniformly bounded from below on K, we conclude that

$$\lim_{N\to\infty} \frac{G_N'}{G_N} = \frac{F'}{F}$$

uniformly on K, and because K is an arbitrary compact subset of Ω, the limit holds for every point of Ω.

- Moreover, a simple calculation yields

$$\frac{G_N'}{G_N} = \sum_{n=1}^{N} \frac{F_n'}{F_n},$$

so part (ii) of the lemma is also proved. □

Canonical factors

- For each integer $k \geq 0$ we define the **canonical factors** by

$$E_0(z) = 1 - z,$$

and

$$E_k(z) = (1 - z)e^{z + z^2/2 + \cdots + z^k/k}, \quad \text{for } k \geq 1.$$

The integer k is called the **degree** of the canonical factor.

Lemma

If $|z| \leq 1/2$, then

$$|E_k(z) - 1| \leq 2e|z|^{k+1}.$$

Canonical factors

Proof: If $|z| \leq 1/2$, then with the logarithm defined in terms of the power series, we have $1 - z = e^{\log(1-z)}$.

- Therefore
$$E_k(z) = e^{\log(1-z)+z+z^2/2+\cdots+z^k/k} = e^w$$
where $w = -\sum_{n=k+1}^{\infty} z^n/n$. Observe that since $|z| \leq 1/2$ we have
$$|w| \leq |z|^{k+1} \sum_{n=k+1}^{\infty} |z|^{n-k-1}/n \leq |z|^{k+1} \sum_{j=0}^{\infty} 2^{-j} \leq 2|z|^{k+1}.$$
- In particular, we have $|w| \leq 1$ and this implies that
$$|1 - E_k(z)| = |1 - e^w| \leq e|w| \leq 2e|z|^{k+1}. \qquad \square$$

Weierstrass infinite products

Theorem

Given any sequence $(a_n)_{n\in\mathbb{Z}_+} \subseteq \mathbb{C}$ with $\lim_{n\to\infty} |a_n| = \infty$, there exists an entire function f that vanishes at all $z = a_n$ and nowhere else. Any other such entire function is of the form $f(z)e^{g(z)}$, where g is entire.

Proof:

- Recall that if a holomorphic function f vanishes at $z = a$, then the multiplicity of the zero a is the integer m so that

$$f(z) = (z-a)^m g(z),$$

where g is holomorphic and nowhere vanishing in a neighborhood of a.

Weierstrass infinite products

- To begin the proof, note first that if f_1 and f_2 are two entire functions that vanish at all $z = a_n$ and nowhere else, then f_1/f_2 has removable singularities at all the points a_n. Hence f_1/f_2 is entire and vanishes nowhere, so that there exists an entire function g with

$$f_1(z)/f_2(z) = e^{g(z)}.$$

- Therefore $f_1(z) = f_2(z)e^{g(z)}$ as desired.
- We have to construct a function that vanishes at all the points of the sequence $(a_n)_{n\in\mathbb{Z}_+}$ and nowhere else.
- Suppose that we are given a zero of order m at the origin, and that $a_1, a_2 \ldots$ are all non-zero. Then we define the **Weierstrass product** by

$$f(z) = z^m \prod_{n=1}^{\infty} E_n\left(z/a_n\right).$$

Weierstrass infinite products

- We claim that this function has the required properties; that is,

(i) f is entire with a zero of order m at the origin;

(ii) f has zeros at each point of the sequence $(a_n)_{n\in\mathbb{Z}_+}$;

(iii) f vanishes nowhere else.

- Fix $R > 0$, and suppose that z belongs to the disc $D(0, R)$. We shall prove that f has all the desired properties in this disc, and since R is arbitrary, this will prove the theorem.
- We can consider two types of factors in the formula defining f, with the choice depending on whether $|a_n| \leq 2R$ or $|a_n| > 2R$.
- There are only finitely many terms of the first kind, since

$$\lim_{n\to\infty} |a_n| = \infty,$$

and we see that the finite product vanishes at all $z = a_n \in D(0, R)$.

Weierstrass infinite products

- If $|a_n| \geq 2R$, we have $|z/a_n| \leq 1/2$, hence the previous lemma implies

$$|E_n(z/a_n) - 1| \leq 2e \left|\frac{z}{a_n}\right|^{n+1} \leq \frac{e}{2^n}.$$

- Therefore, the product

$$\prod_{|a_n| \geq 2R} E_n(z/a_n)$$

  defines a holomorphic function when $|z| < R$, and does not vanish in that disc by the previous lemmas.
- This shows that the function f has the desired properties, and the proof of Weierstrass's theorem is complete. □

Hadamard's theorem

Theorem

Suppose f is entire and has growth order ρ_0. Let $k \in \mathbb{Z}$ be so that $k \leq \rho_0 < k+1$. If $a_1, a_2, \ldots$ denote the (non-zero) zeros of f, then

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(z/a_n\right),$$

where P is a polynomial of degree $\leq k$, and m is the order of the zero of f at $z=0$, and E_k are the canonical factors for $k \in \mathbb{N}$.

- We gather a few lemmas needed in the proof of Hadamard's theorem.

Hadamard's theorem

Lemma

The canonical products satisfy

$$|E_k(z)| \geq e^{-c|z|^{k+1}} \quad \text{if} \quad |z| \leq 1/2,$$

and

$$|E_k(z)| \geq |1 - z| e^{-c|z|^k} \quad \text{if} \quad |z| \geq 1/2.$$

Here, we allow the implied constant $c = c_k$ to depend on $k \in \mathbb{N}$.

Proof:

- If $|z| \leq 1/2$ we can use the power series to define the logarithm of $1 - z$, so that

$$E_k(z) = e^{\log(1-z) + \sum_{n=1}^{k} z^n/n} = e^{-\sum_{n=k+1}^{\infty} z^n/n} = e^w.$$

Since $|e^w| \geq e^{-|w|}$ and $|w| \leq c|z|^{k+1}$, the first part follows.

Hadamard's theorem

- For the second part, simply observe that if $|z| \geq 1/2$, then

$$|E_k(z)| = |1 - z| \left| e^{z + z^2/2 + \cdots + z^k/k} \right|,$$

and that there exists $c' > 0$ such that

$$\left| e^{z + z^2/2 + \cdots + z^k/k} \right| \geq e^{-\left| z + z^2/2 + \cdots + z^k/k \right|} \geq e^{-c'|z|^k}. \qquad \square$$

Lemma

For any $s \in \mathbb{R}_+$ with $\rho_0 < s < k + 1$, we have

$$\left| \prod_{n=1}^{\infty} E_k\left(z/a_n\right) \right| \geq e^{-c|z|^s},$$

except possibly when z belongs to the union of the discs centered at a_n of radius $|a_n|^{-k-1}$, for $n \in \mathbb{Z}_+$.

Hadamard's theorem

Proof: First, we write

$$\prod_{n=1}^{\infty} E_k\left(z/a_n\right) = \prod_{|a_n|\leq 2|z|} E_k\left(z/a_n\right) \prod_{|a_n|>2|z|} E_k\left(z/a_n\right)$$

- For the second product the estimate asserted above holds for all $z \in \mathbb{C}$. Indeed, by the previous lemma

$$\left|\prod_{|a_n|>2|z|} E_k\left(z/a_n\right)\right| = \prod_{|a_n|>2|z|} \left|E_k\left(z/a_n\right)\right|$$
$$\geq \prod_{|a_n|>2|z|} e^{-c|z/a_n|^{k+1}} \geq e^{-c|z|^{k+1}\sum_{|a_n|>2|z|}|a_n|^{-k-1}}.$$

- But $|a_n| > 2|z|$ and $s < k+1$, so we must have

$$|a_n|^{-k-1} = |a_n|^{-s}\,|a_n|^{s-k-1} \leq C\,|a_n|^{-s}\,|z|^{s-k-1}.$$

Hadamard's theorem

- Therefore, the fact that $\sum_{n\in\mathbb{Z}_+} |a_n|^{-s}$ converges implies that

$$\left|\prod_{|a_n|>2|z|} E_k(z/a_n)\right| \geq e^{-c|z|^s}$$

for some constant $c > 0$, which may depend on k and s.

- To estimate the first product, we use the second part of the previous lemma, and write

$$\left|\prod_{|a_n|\leq 2|z|} E_k(z/a_n)\right| \geq \prod_{|a_n|\leq 2|z|} \left|1 - \frac{z}{a_n}\right| \prod_{|a_n|\leq 2|z|} e^{-c|z/a_n|^k}. \qquad (*)$$

- We now note that

$$\prod_{|a_n|\leq 2|z|} e^{-c|z/a_n|^k} = e^{-c|z|^k \sum_{|a_n|\leq 2|z|} |a_n|^{-k}},$$

and again, we have $|a_n|^{-k} = |a_n|^{-s} |a_n|^{s-k} \leq C |a_n|^{-s} |z|^{s-k}$.

Hadamard's theorem

- This proves that
$$\prod_{|a_n|\le 2|z|} e^{-c'|z/a_n|^k} \ge e^{-c|z|^s}.$$
- The estimate on the first product on the right-hand side of (*) will require the restriction on z imposed in the statement of the lemma.
- Indeed, whenever z does not belong to a disc of radius $|a_n|^{-k-1}$ centered at a_n, we must have $|a_n - z| \ge |a_n|^{-k-1}$.
- Therefore
$$\prod_{|a_n|\le 2|z|}\left|1-\frac{z}{a_n}\right| = \prod_{|a_n|\le 2|z|}\left|\frac{a_n - z}{a_n}\right| \ge \prod_{|a_n|\le 2|z|} |a_n|^{-k-1}\,|a_n|^{-1} = \prod_{|a_n|\le 2|z|} |a_n|^{-k-2}.$$

Hadamard's theorem

- Finally, the estimate for the first product follows from the fact that

$$(k+2)\sum_{|a_n|\leq 2|z|}\log|a_n| \leq (k+2)\mathfrak{n}_f(2|z|)\log 2|z|$$
$$\leq c|z|^s \log 2|z|$$
$$\leq c'|z|^{s'}$$

for any $s' > s$, and the second inequality follows as $\mathfrak{n}(2|z|) \leq c|z|^s$.

- Hence, we have proved that

$$\left|\prod_{n=1}^{\infty} E_k\left(z/a_n\right)\right| \geq e^{-c'|z|^{s'}},$$

except possibly when z belongs to the union of the discs centered at a_n of radius $|a_n|^{-k-1}$, for $n \in \mathbb{Z}_+$.

- Since we restricted s to satisfy $s > \rho_0$, we can take an initial s sufficiently close to ρ_0, so that the assertion of the lemma is established (with s being replaced by s'). □

Hadamard's theorem

Corollary

There exists a sequence of radii, $r_1, r_2, \ldots$, with $r_m \to \infty$, such that

$$\left| \prod_{n=1}^{\infty} E_k(z/a_n) \right| \geq e^{-c|z|^s} \quad \textit{for} \quad |z| = r_m.$$

Proof:

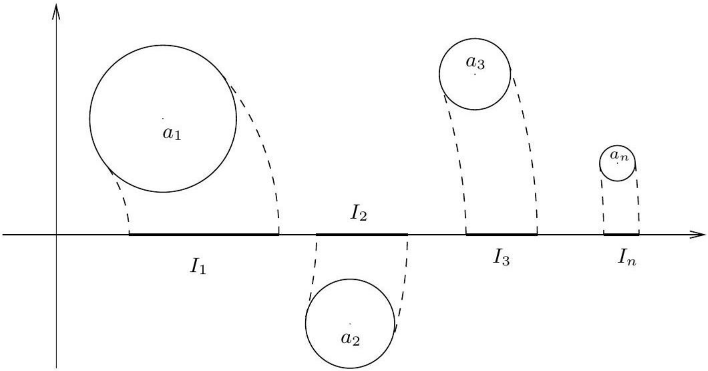
- Since $\sum_{n \in \mathbb{Z}_+} |a_n|^{-k-1} < \infty$, there exists $N \in \mathbb{Z}_+$ so that

$$\sum_{n=N}^{\infty} |a_n|^{-k-1} < 1/10.$$

- Therefore, given any two consecutive large integers L and $L+1$, we can find $r \in \mathbb{R}_+$ with $L \leq r \leq L+1$, such that the circle of radius r centered at the origin does not intersect the forbidden discs from the previous lemma.

Hadamard's theorem

- Otherwise, the union of the intervals $I_n = \left[\,|a_n| - \frac{1}{|a_n|^{k+1}}, |a_n| + \frac{1}{|a_n|^{k+1}}\right]$ (which are of length $2\,|a_n|^{-k-1}$) would cover all the interval $[L, L+1]$.



- This would imply $2\sum_{n=N}^{\infty} |a_n|^{-k-1} \geq 1$, which is a contradiction. We then apply the previous lemma with $|z| = r$ and the proof follows. □

Proof of Hadamard's theorem

Proof: Let

$$E(z) = z^m \prod_{n=1}^{\infty} E_k\left(z/a_n\right).$$

- To prove that E is entire, we repeat the argument from the proof of the Weierstrass theorem. Namely, we have

$$\left|E_k\left(z/a_n\right) - 1\right| \leq 2e\left|\frac{z}{a_n}\right|^{k+1}, \quad \text{for all large} \quad n \in \mathbb{Z}_+,$$

  and that the series $\sum_{n\in\mathbb{Z}_+} |a_n|^{-k-1}$ converges. (Here, recall that $\rho_0 < s < k+1$.)
- Moreover, E has the zeros of f, therefore f/E is holomorphic and nowhere vanishing. Hence

$$\frac{f(z)}{E(z)} = e^{g(z)}$$

  for some entire function g.

Proof of Hadamard's theorem

- By the fact that f has growth order ρ_0, and because of the estimate from below for E obtained in the previous corollary, we have

$$e^{\mathrm{Re}(g(z))} = \left|\frac{f(z)}{E(z)}\right| \leq c' e^{c|z|^s}, \quad \text{whenever} \quad |z| = r_m.$$

- This proves that

$$\mathrm{Re}(g(z)) \leq C|z|^s, \quad \text{for} \quad |z| = r_m,$$

where $(r_m)_{m\in\mathbb{Z}_+} \subseteq \mathbb{R}_+$ is a a sequence such that $\lim_{m\to\infty} r_m = \infty$.

- We have to prove that g is a polynomial of degree $\leq s$.
- We can expand g in a power series centered at the origin

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

Proof of Hadamard's theorem

- As a simple application of Cauchy's integral formulas, we may write

$$\frac{1}{2\pi}\int_0^{2\pi} g\left(re^{i\theta}\right)e^{-in\theta}d\theta = \begin{cases} a_n r^n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

- By taking complex conjugates we find that

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{g\left(re^{i\theta}\right)}e^{-in\theta}d\theta = 0$$

whenever $n > 0$.

- Since $2u = g + \overline{g}$ we add the above two equations and obtain

$$a_n r^n = \frac{1}{\pi}\int_0^{2\pi} u\left(re^{i\theta}\right)e^{-in\theta}d\theta, \qquad \text{whenever} \qquad n > 0.$$

- For $n = 0$ we find that

$$2\,\mathrm{Re}\left(a_0\right) = \frac{1}{\pi}\int_0^{2\pi} u\left(re^{i\theta}\right)d\theta.$$

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- Now we recall the simple fact that whenever $n \neq 0$, the integral of $e^{-in\theta}$ over any circle centered at the origin vanishes. Therefore

$$a_n = \frac{1}{\pi r^n} \int_0^{2\pi} \left[u\left(re^{i\theta} \right) - Cr^s \right] e^{-in\theta} d\theta \quad \text{when} \quad n > 0.$$

- Taking $r = r_m$, we consequently obtain

$$|a_n| \leq \frac{1}{\pi r_m^n} \int_0^{2\pi} \left[Cr_m^s - u\left(r_m e^{i\theta} \right) \right] d\theta \leq 2Cr_m^{s-n} - 2\,\mathrm{Re}\left(a_0 \right) r_m^{-n}.$$

- Letting $m \to \infty$ we deduce $a_n = 0$ for any $n > s$.
- This completes the proof of Hadamard's theorem. □