Lecture 20

Introduction to the prime number theorem
Analytic continuation of the Riemann zeta function

MATH 503, FALL 2025

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Important arithmetic functions involvong primes

1. The **von Mangoldt function** Λ is defined by

$$\Lambda(n) := \begin{cases} \log p, & \text{if } n = p^k \text{ for some prime } p \text{ and } k \in \mathbb{Z}_+, \\ 0, & \text{otherwise} \end{cases}.$$

2. The **first Chebyshev function** ϑ is defined for $x \geqslant 2$ by

$$\vartheta(x) := \sum_{p \in \mathbb{P}_{\leq x}} \log p,$$

while it is convenient to set $\vartheta(x) := 0$ for $0 < x < 2$, where

$$\mathbb{P}_{\leq x} := \{p \in \mathbb{P} : p \leq x\} = \mathbb{P} \cap [0, x].$$

Important arithmetic functions involvong primes

3. The **second Chebyshev function** ψ is defined for $x \geqslant 2$ by

$$\psi(x) := \sum_{n \in [x]} \Lambda(n),$$

while it is convenient to set $\psi(x) := 0$ for $0 < x < 2$.

4. The **prime counting function** π is defined by

$$\pi(x) := \sum_{p \in \mathbb{P}_{\leq x}} 1 = \#\mathbb{P}_{\leq x},$$

while it is convenient to set $\pi(x) := 0$ for $0 < x < 2$.

Simple relations

Theorem

For $x \geq 2$ we have

$$\vartheta(x) = \pi(x)\log x - \int_2^x \pi(t)\frac{dt}{t},$$

and

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t\log^2 t}dt.$$

Proof: We have

$$\pi(x) = \sum_{p \in \mathbb{P}_{\leq x}} 1 = \sum_{1<n\leq x} \mathbf{1}_{\mathbb{P}}(n),$$

and

$$\vartheta(x) = \sum_{p \in \mathbb{P}_{\leq x}} \log p = \sum_{1<n\leq x} \mathbf{1}_{\mathbb{P}}(n)\log n.$$

Simple relations

- If $x, y \in \mathbb{R}_+$ with $\lfloor y \rfloor < \lfloor x \rfloor$, and $g \in C^1([y, x])$, then we know

$$\sum_{y<n\leq x} f(n)g(n) = F(x)g(x) - F(y)g(y) - \int_y^x F(t)g'(t)dt,$$

where $F(t) := \sum_{1\leq n\leq x} f(n)$.

- Taking $f(n) = \mathbf{1}_{\mathbb{P}}(n)$ and $g(x) = \log x$ with $y = 1$ we obtain

$$\vartheta(x) = \sum_{1<n\leq x} \mathbf{1}_{\mathbb{P}}(n) \log n = \pi(x) \log x - \pi(1) \log 1 - \int_1^x \pi(t)\frac{dt}{t},$$

which proves the first identity since $\pi(t) = 0$ for $t < 2$.

Simple relations

- Next, let $f(n) = \mathbf{1}_{\mathbb{P}}(n) \log n$ and $g(x) = 1/\log x$ and write

$$\pi(x) = \sum_{3/2<n\leq x} f(n)\frac{1}{\log n}, \quad \vartheta(x) = \sum_{1<n\leq x} f(n)$$

- Using the summation by parts formula with $y = 3/2$ we obtain

$$\pi(x) = \frac{\vartheta(x)}{\log x} - \frac{\vartheta(3/2)}{\log 3/2} + \int_{3/2}^{x} \frac{\vartheta(t)}{t\log^2 t} dt$$

which proves the second identity, since $\vartheta(t) = 0$ if $t < 2$. $\square$

Pointwise bounds

Lemma

(i) *For all* $x \in \mathbb{R}_+$, *we have*

$$\vartheta(x) \leqslant \psi(x) \leqslant \vartheta(x) + \pi(\sqrt{x}) \log x.$$

(ii) *For all* $x \geqslant 2$ *and all* $a > 1$, *we have*

$$\frac{\vartheta(x)}{\log x} \leqslant \pi(x) \leqslant \frac{a\vartheta(x)}{\log x} + \pi\left(x^{1/a}\right).$$

Proof of (i): One may suppose $x \geqslant 2$. We first have

$$\psi(x) - \vartheta(x) = \sum_{p^k \leqslant x} \log p - \sum_{p \leqslant x} \log p = \sum_{p \leqslant \sqrt{x}} \sum_{k=2}^{\lfloor \frac{\log x}{\log p} \rfloor} \log p,$$

so that $\psi(x) \geqslant \vartheta(x)$.

Pointwise bounds

- On the other hand, we have

$$\psi(x) - \vartheta(x) = \sum_{p \leqslant \sqrt{x}} \sum_{k=2}^{\lfloor \frac{\log x}{\log p} \rfloor} \log p \leqslant \sum_{p \leqslant \sqrt{x}} \log p \left\lfloor \frac{\log x}{\log p} \right\rfloor \leqslant \sum_{p \leqslant \sqrt{x}} \log x$$
$$= \pi(\sqrt{x}) \log x. \qquad \square$$

Proof of (ii): We have

$$\pi(x) = \sum_{p \leqslant x} 1 = \sum_{p \leqslant x} \frac{\log p}{\log p} \geqslant \frac{1}{\log x} \sum_{p \leqslant x} \log p = \frac{\vartheta(x)}{\log x}.$$

- For $2 \leqslant T < x$, we also have

$$\pi(x) = \sum_{p \leqslant T} 1 + \sum_{T < p \leqslant x} 1 = \pi(T) + \sum_{T < p \leqslant x} \frac{\log p}{\log p} \leqslant \pi(T) + \frac{\vartheta(x)}{\log T}.$$

and the choice of $T = x^{1/a}$ implies the asserted estimate. $\square$

Equivalent forms of the prime number theorem

Theorem

The following relations are equivalent:

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1. \tag{A}$$

$$\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1. \tag{B}$$

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1. \tag{C}$$

Proof: We know that

$$\frac{\vartheta(x)}{x} \leqslant \frac{\psi(x)}{x} \leqslant \frac{\vartheta(x)}{x} + \frac{\pi(\sqrt{x})\log x}{x}.$$

Hence the equivalence between (B) and (C) follows, since

$$\lim_{x\to\infty} \frac{\pi(\sqrt{x})\log x}{x} = 0.$$

Equivalent forms of the prime number theorem

- For every $a > 1$ know that

$$\frac{\vartheta(x)}{x} \leqslant \frac{\pi(x)\log x}{x} \leqslant \frac{a\vartheta(x)}{x} + \frac{\pi\left(x^{1/a}\right)\log x}{x}.$$

- For every $a > 1$ we also know that

$$\lim_{x\to\infty} \frac{\pi\left(x^{1/a}\right)\log x}{x} = 0.$$

- Then

$$\lim_{x\to\infty} \frac{\vartheta(x)}{x} \leqslant \lim_{x\to\infty} \frac{\pi(x)\log x}{x} \leqslant \lim_{x\to\infty} \frac{a\vartheta(x)}{x}.$$

- Since $a > 1$ is arbitrary, we obtain equivalence between (A) and (B).
- This completes the proof of the theorem. □

Chebyshev theorem, upper bound

Theorem

For all $x \geqslant 1$, we have

$$\vartheta(x) \leqslant x \log 4.$$

Proof: We first prove by induction that, for all $n \in \mathbb{Z}_+$, we have

$$\vartheta(n) \leqslant n \log 4.$$

- This inequality is clearly true for $n \in [3]$. If $n \geqslant 4$ is even, we have

$$\vartheta(n) = \vartheta(n-1) \leqslant (n-1) \log 4 < n \log 4^n.$$

- Suppose now that $n \geqslant 5$ is odd and set $n = 2m + 1$ with $m \in \mathbb{Z}_+$. The idea is to use the fact that the product

$$\prod_{m+1<p\leqslant 2m+1} p \qquad \text{divides the binomial coefficient} \qquad \binom{2m+1}{m}.$$

Chebyshev theorem, upper bound

- To see this, observe that $p \in \mathbb{P}$ such that $m+1 < p \leqslant 2m+1$ divides $(2m+1)!$ because of $p \leqslant 2m+1$, but does not divide $m!(m+1)!$ because of $p > m+1$, so that

$$\prod_{m+1<p\leqslant 2m+1} p \quad \text{divides} \quad (2m+1)! = m!(m+1)!\binom{2m+1}{m}$$

  and since the product is coprime to $m!(m+1)!$ the claim follows.
- Taking logarithms, we then obtain

$$\vartheta(2m+1) - \vartheta(m+1) = \sum_{m+1<p\leqslant 2m+1} \log p \leqslant \log \binom{2m+1}{m}.$$

- Using Stirling's formula we have $\binom{2m+1}{m} \leq \frac{2m+1}{m+1}\frac{4^m}{\sqrt{\pi m}} \leq 4^m$, thus

$$\vartheta(2m+1) \leq m\log 4 + \vartheta(m+1) \leq m\log 4 + (m+1)\log 4 = (2m+1)\log 4,$$

  where we have used the induction hypothesis applied to $\vartheta(m+1)$.

Chebyshev theorem, lower bound

- The lemma follows from

$$\vartheta(x) = \vartheta(\lfloor x \rfloor) \leq \lfloor x \rfloor \log 4 \leqslant x \log 4.$$

- The proof for the upper bound is now completed. □

Theorem

For all $x \geqslant 1537$*, we have*

$$\vartheta(x) > \frac{x}{\log 4}.$$

Proof: We first notice that the function f defined by

$$f(x) = \lfloor x \rfloor - \left\lfloor \frac{x}{2} \right\rfloor - \left\lfloor \frac{x}{3} \right\rfloor - \left\lfloor \frac{x}{5} \right\rfloor + \left\lfloor \frac{x}{30} \right\rfloor,$$

is periodic of period 30, since $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ for any $n \in \mathbb{Z}$.

Chebyshev theorem, lower bound

- Moreover, for $x \notin \mathbb{Z}$, we have
$$f(30-x) = 1 - f(x),$$
since $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$ for $x \notin \mathbb{Z}$.
- An inspection of its values when $x \in [1, 15)$ allows us to infer that $f(x)$ only takes the values 0 or 1 if $x \notin \mathbb{Z}$.
- Since f is continuous on the right, we also have $f(x) = 0$ or 1 when $x \in \mathbb{Z}$. By periodicity, we infer that $f(x) = 0$ or 1 for all $x \in \mathbb{R}$.
- Since $\lfloor x/kn \rfloor = 0$ whenever $n > x/k$ for $k \in \{2, 3, 5, 30\}$, we obtain

$$\begin{aligned}\psi(x) \geqslant \sum_{n \leqslant x} \Lambda(n) f\left(\frac{x}{n}\right) &= \sum_{n \leqslant x} \Lambda(n) \left(\left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{2n} \right\rfloor - \left\lfloor \frac{x}{3n} \right\rfloor - \left\lfloor \frac{x}{5n} \right\rfloor + \left\lfloor \frac{x}{30n} \right\rfloor \right) \\ &= \sum_{n \leqslant x} \Lambda(n) \left\lfloor \frac{x}{n} \right\rfloor - \sum_{n \leqslant x/2} \Lambda(n) \left\lfloor \frac{x}{2n} \right\rfloor - \sum_{n \leqslant x/3} \Lambda(n) \left\lfloor \frac{x}{3n} \right\rfloor \\ &\quad - \sum_{n \leqslant x/5} \Lambda(n) \left\lfloor \frac{x}{5n} \right\rfloor + \sum_{n \leqslant x/30} \Lambda(n) \left\lfloor \frac{x}{30n} \right\rfloor.\end{aligned}$$

Chebyshev theorem, lower bound

- By a simplified form of Striling's formula we know for all $x \geq 1$ that

$$\sum_{n \in [x]} \log n = x \log x - x + 1 + R(x), \quad \text{where} \quad 0 \leq R(x) \leq \log x.$$

- Since $\Lambda \star \mathbf{1} = \log$, then we conclude that

$$\sum_{n \in [x]} \log n = \sum_{n \in [x]} \sum_{d | n} \Lambda(d) = \sum_{d \in [x]} \Lambda(d) \sum_{k \in [x/d]} = \sum_{d \in [x]} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor.$$

- Hence for $x \geq 1$, we have

$$\sum_{d \in [x]} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor = x \log x - x + 1 + R(x), \quad \text{where} \quad 0 \leq R(x) \leq \log x.$$

Chebyshev theorem, lower bound

- Inserting this bounds to the previous formula we obtain

$$\begin{aligned}\psi(x) \geqslant & x\log x - x + 1 - \left(\frac{x}{2}\log\frac{x}{2} - \frac{x}{2} + 1 + \log\frac{x}{2}\right) \\ & - \left(\frac{x}{3}\log\frac{x}{3} - \frac{x}{3} + 1 + \log\frac{x}{3}\right) - \left(\frac{x}{5}\log\frac{x}{5} - \frac{x}{5} + 1 + \log\frac{x}{5}\right) \\ & + \left(\frac{x}{30}\log\frac{x}{30} - \frac{x}{30} + 1\right) \\ = & x\log\left(2^{7/15}3^{3/10}5^{1/6}\right) - 3\log x + \log 30 - 1.\end{aligned}$$

- Using the estimate $\psi(x) \leq \vartheta(x) + \pi(\sqrt{x})\log x$ we obtain the desired lower bound

$$\vartheta(x) \geq x\log\left(2^{7/15}3^{3/10}5^{1/6}\right) - (\sqrt{x} + 3)\log x \geq \frac{x}{\log 4},$$

whenever $x \in \mathbb{R}_+$ is sufficiently large. This completes the proof. □

Bertrand's postulate

Note that

$$\log\left(2^{7/15}3^{3/10}5^{1/6}\right) \approx 0.92129\ldots,$$

which is a very good lower bound. It was sufficient to allow Chebyshev to prove Bertrand's famous postulate.

Theorem

Let $n \in \mathbb{Z}_+$. Then the interval $(n, 2n]$ contains a prime number.

Proof: We check numerically the result for $n \in [768]$.

- Suppose $n \geqslant 769$, using Chebyshev's upper and lower bounds we deduce

$$\sum_{n<p\leqslant 2n} \log p = \vartheta(2n) - \vartheta(n) > n\left(\frac{2}{\log 4} - \log 4\right) > 0.$$

- This shows that $(n, 2n] \cap \mathbb{P} \neq \emptyset$, which implies the desired result. $\square$

Euler–Maclaurin–Jacobi summation formula

Theorem

Let $b > a$ and $q \geq 1$ be integers. Let $f \in C^q([a, b])$, then

$$\sum_{n=a+1}^{b} f(n) = \int_a^b f(x)dx + \sum_{r=1}^{q} (-1)^r \frac{B_r}{r!} \left(f^{(r-1)}(b) - f^{(r-1)}(a) \right) + R_q,$$

where

$$R_q = \frac{(-1)^{q+1}}{q!} \int_a^b B_q(x - \lfloor x \rfloor) f^{(q)}(x) dx.$$

Euler's summation formula

Theorem

If $f \in C^1([a,b])$, *and* $\psi(x) = \{x\} - 1/2$ *for* $x \in \mathbb{R}$, *then*

$$\sum_{a<n\leqslant b} f(n) = \int_a^b f(x)dx + f(a)\psi(a) - f(b)\psi(b) + \int_a^b f'(x)\psi(x)dx.$$

Proof: We apply the Euler–Maclaurin–Jacobi summation formula with $q = 1$, then

$$B_1(x) = B_0 x + B_1 = x - \frac{1}{2}, \quad \text{and} \quad B_1'(x) = 1,$$

and consequently, we have

$$R_1 = \int_a^b f'(x)\psi(x)dx,$$

which completes the proof. □

Riemann zeta-function

Definition

The **Riemann zeta-function** $\zeta(s)$ is defined for all complex numbers $s = \sigma + it$ such that $\sigma > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

- By the absolute convergence for all complex numbers $s = \sigma + it$ such that $\sigma > 1$ we also have the Euler product formula

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

- The Euler product formula enables us to see that $\zeta(s) \neq 0$ in the half-plane $\sigma > 1$.

Remarks

- Indeed, for $\sigma > 1$ we have

$$\frac{1}{|\zeta(s)|} = \prod_{p\in\mathbb{P}} \left|1 - \frac{1}{p^s}\right| \leq \prod_{p\in\mathbb{P}} \left(1 + \frac{1}{p^\sigma}\right)$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq 1 + \int_1^\infty \frac{dt}{t^\sigma} = \frac{\sigma}{\sigma - 1}.$$

Thus $|\zeta(s)| \geq \frac{\sigma-1}{\sigma} > 0$.

Euler's summation formula

If $f \in C^1([a,b])$, and $\psi(x) = \{x\} - 1/2$ for $x \in \mathbb{R}$, then

$$\sum_{a<n\leqslant b} f(n) = \int_a^b f(x)dx + f(a)\psi(a) - f(b)\psi(b) + \int_a^b f'(x)\psi(x)dx.$$

Riemann zeta-function

- Let $x \geqslant 1$ be a real number and $s = \sigma + it$ with $\sigma > 1$. By the Euler summation formula with $a = 1, b = x$ and $f(x) = x^{-s}$, we can write

$$\sum_{n \leqslant x} \frac{1}{n^s} = \frac{1}{2} + \frac{1 - x^{1-s}}{s-1} - \frac{\psi(x)}{x^s} - s \int_1^x \frac{\psi(u)}{u^{s+1}} du.$$

- Taking $x \to \infty$ we obtain

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} - s \int_1^\infty \frac{\psi(u)}{u^{s+1}} du. \qquad (*)$$

- Since $|\psi(x)| \leqslant \frac{1}{2}$, the integral converges for $\operatorname{Re} s > 0$ and is uniformly convergent in any compact set to the right of the line $\operatorname{Re} s = 0$.
- This implies that it defines an analytic function in the half-plane $\operatorname{Re} s > 0$, and therefore (*) extends ζ to a meromorphic function in this half-plane, which is analytic except for a simple pole at $s = 1$ with residue 1.

Theta function

- Replacing x by $\pi n^2 x$ in the integral defining $\Gamma(s/2)$ gives
$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty x^{s/2-1} e^{-\pi n^2 x} dx \quad \text{for all} \quad \sigma > 0.$$
- The purpose is to sum both sides of this equation. To this end, we define the following two Theta functions. For all $x > 0$, we set
$$\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x} \quad \text{and} \quad \theta(x) = 2\omega(x) + 1 = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}.$$
- Then $g(t) = e^{-\pi t^2}$ satisfies $\int_{\mathbb{R}} g(t)dt = 1$, and
$$\widehat{g}(u) = e^{-\pi u^2}.$$
- For a Schwartz function f, by the Poisson summation formula, we have $\sum_{n\in\mathbb{Z}} \widehat{f}(n) = \sum_{n\in\mathbb{Z}} f(n)$, hence
$$\theta(x) = \sum_{n\in\mathbb{Z}} g(\sqrt{x}n) = x^{-1/2}\theta(x^{-1}) \quad \text{for all} \quad x > 0.$$

Theta function

- Summing this equation over $n \in \mathbb{Z}_+$ and interchanging the sum and integral, we obtain for all $\sigma > 1$ that

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{s/2-1}\omega(x)dx,$$

since the sum and integral converge absolutely for $\sigma > 1$.

- Splitting the integral $\int_0^\infty = \int_0^1 + \int_1^\infty$ and changing the variables $x \mapsto 1/x$ in the first integral yields

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^\infty x^{s/2-1}\omega(x)dx + \int_1^\infty x^{-s/2-1}\omega\left(\frac{1}{x}\right)dx.$$

- Using $\theta(x^{-1}) = x^{1/2}\theta(x)$ we write $\omega\left(\frac{1}{x}\right) = x^{1/2}\omega(x) + \frac{x^{1/2}-1}{2}$, and

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \omega(x)\left(x^{s/2} + x^{(1-s)/2}\right)\frac{dx}{x},$$

whenever $\sigma > 1$.

Functional equation

Theorem

Let

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$
$$= 1 + \frac{s(s-1)}{2}\int_1^\infty (\theta(x)-1)\left(x^{s/2} + x^{(1-s)/2}\right)\frac{dx}{x},$$

where θ is the Theta function $\theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$.

- *Then the function $\xi(s)$ can be extended analytically in the whole complex plane to an entire function that satisfies the functional equation $\xi(s) = \xi(1-s)$.*
- *Thus the Riemann zeta-function can be extended analytically in the whole complex plane to a meromorphic function having a simple pole at $s = 1$ with residue 1. Furthermore, for all $s \in \mathbb{C}\backslash\{1\}$, we have*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

Functional equation

Proof: For $\sigma > 1$ we have

$$\Xi(s) = -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \omega(x)\left(x^{s/2} + x^{(1-s)/2}\right)\frac{dx}{x}. \qquad (*)$$

Then we see that

$$\xi(s) = s(s-1)\Xi(s).$$

- Since $\omega(x) = O(e^{-\pi x})$ as $x \to \infty$, we infer that the integral in (*) is absolutely convergent for all $s \in \mathbb{C}$ whereas the left-hand side is a meromorphic function on $\sigma > 0$. This implies that:

(i) The identity (*) is valid for all $\sigma > 0$.

(ii) The function $\Xi(s)$ can be defined by this identity as a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0$ and $s = 1$.

(iii) Since the right-hand side of (*) is invariant under the substitution $s \mapsto 1 - s$, we obtain $\Xi(s) = \Xi(1-s)$ and $\xi(s) = \xi(1-s)$.

(iv) The function $s \mapsto \xi(s) = s(s-1)\Xi(s)$ is entire on $\mathbb{C}$. Indeed, if $\sigma > 0$, the factor $s - 1$ counters the pole at $s = 1$, and the result on all $\mathbb{C}$ follows from the functional equation.

Functional equation

- It remains to show that the functional equation can be written as

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

- Since $\Xi(s) = \Xi(1-s)$, we have

$$\Gamma(s/2)\zeta(s) = \pi^{s/2}\Xi(s) = \pi^{s/2}\Xi(1-s) = \pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

- Multiplying both sides by $\pi^{-1/2}2^{s-1}\Gamma\left(\frac{1+s}{2}\right)$ and using the duplication formula, asserting that $\Gamma(s) = \pi^{-1/2}2^{s-1}\Gamma\left(s/2\right)\Gamma\left((s+1)/2\right)$ we see

$$\Gamma(s)\zeta(s) = (2\pi)^{s-1}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\zeta(1-s)$$

Functional equation

- Now the reflection formula $\frac{\sin \pi s}{\pi} = \frac{1}{\Gamma(s)\Gamma(1-s)}$, implies that

$$\zeta(s) = (2\pi)^{s-1} \left(\frac{\sin \pi s}{\sin(\pi(1+s)/2)} \right) \Gamma(1-s)\zeta(1-s)$$

and the result follows from the identity

$$\sin \pi s = 2 \sin \left(\frac{\pi s}{2} \right) \sin \left(\frac{\pi}{2}(1+s) \right).$$

- The proof is complete. □

Remarks

- $\zeta(s)$ has simple zeros at $s = -2, -4, -6, -8, \ldots$. Indeed, since the integral in (*) is absolutely convergent for all $s \in \mathbb{C}$ and since $\omega(x) > 0$ for all $x \in \mathbb{R}$, we have

$$\Xi(-2n) = \frac{1}{2n} - \frac{1}{2n+1} + \int_1^\infty \omega(x) \left(x^{-n} + x^{n+1/2} \right) \frac{dx}{x} > 0$$

  for all $n \in \mathbb{Z}_+$. The result follows from the fact that $\Gamma(s/2)$ has simple poles at $s = -2n$.
- These zeros are the only ones lying in the region $\sigma < 0$. They are called **trivial zeros** of the Riemann zeta-function.

Remarks

- For all $0 < \sigma < 1$, we have $\zeta(\sigma) \neq 0$. Indeed, for all $\sigma > 0$ we see

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx$$

we infer that, for all $0 < \sigma < 1$, we obtain

$$\left|\zeta(\sigma) - \frac{\sigma}{\sigma - 1}\right| < \sigma \int_1^\infty \frac{dx}{x^{\sigma+1}} = 1,$$

which implies that $\zeta(\sigma) < 1 + \sigma/(\sigma - 1)$ for all $0 < \sigma < 1$.

- Hence $\zeta(\sigma) < 0$ for all $\frac{1}{2} \leqslant \sigma < 1$, and the functional equation implies the asserted result.

Stirling formula

Theorem

Let $0 < \delta < \pi$, *then for any* $z \in \mathbb{C}$ *so that* $|\arg z| < \pi - \delta$, *we have*

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log 2\pi + O(|z|^{-1}),$$

uniformly as $|z| \to \infty$, *where logarithm has principal value, and the implicit constant depend at most on* δ.

Stirling formula

Corollary

Let $a, b \in \mathbb{R}$ *be fixed and* $a \leq b$*.*

(i) *Then for every* $z = \sigma + it$ *with* $\sigma \in [a, b]$ *and* $|t| \geq 1$*, we have*

$$\Gamma(z) = \sqrt{2\pi} e^{-\frac{\pi}{2}|t|} |t|^{\sigma - \frac{1}{2}} e^{i|t|(\log|t| - 1)} e^{\frac{\pi i}{2}(\sigma - \frac{1}{2})} \left(1 + O\left(\frac{1}{|t|}\right)\right).$$

(ii) *Moreover,* $|\Gamma(z)| = \sqrt{2\pi} e^{-\frac{\pi}{2}|t|} |t|^{\sigma - \frac{1}{2}} \left(1 + O\left(\frac{1}{|t|}\right)\right)$.

(iii) *This implies* $|\Gamma(z)| = O\left(e^{-\frac{\pi}{2}|t|} |t|^{\sigma - \frac{1}{2}}\right)$ *and*

(iv) $\frac{1}{|\Gamma(z)|} = O\left(e^{\frac{\pi}{2}|t|} |t|^{\frac{1}{2} - \sigma}\right)$.

Order of ξ

Lemma

The function $\xi(s)$ is an entire function of order 1. Furthermore,

$$\limsup_{|s|\to\infty} \frac{\log|\xi(s)|}{|s|\log|s|} = \frac{1}{2}.$$

Proof: Since $\xi(s) = \xi(1-s)$, it suffices to bound $|\xi(s)|$ for $\mathrm{Re}(s) \geq 1/2$.

- We can estimate $\xi(s)$ by invoking Stirling's formula, and elementary upper bounds for $\zeta(s)$. When $\mathrm{Re}(s) > 1$, we will use the identity

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\,dx.$$

- However, since the last integral represents a holomorphic function in $\mathrm{Re}(s) > 0$, the identity holds in this larger domain.
- In particular, we have

$$|(s-1)\zeta(s)| = O(|s|^2) \quad \text{whenever} \quad \mathrm{Re}(s) \geq 1/2.$$

Proof

- Moreover, since $\log|\Gamma(s)| \le |\log \Gamma(s)|$, Stirling's formula yields

$$\log|\Gamma(s)| \le |s|\log|s| + O(|s|) \quad \text{whenever} \quad \mathrm{Re}(s) \ge 1/2.$$

- Since $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, we obtain

$$\log|\xi(s)| \le \frac{1}{2}|s|\log|s| + O(|s|) \quad \text{whenever} \quad \mathrm{Re}(s) \ge 1/2.$$

  which establishes the first claim of the lemma.

- For the second claim, we note that

$$\log|\Gamma(|s|)| = \log\Gamma(|s|) = |s|\log|s| + O(|s|),$$

  whence

$$\log \xi(|s|) = \frac{1}{2}|s|\log|s| + O(|s|) \quad \text{as} \quad |s| \to \infty.$$

This completes the proof of the lemma. □

Zeros of function ξ

Theorem

The function $\xi(s)$ has infinitely many zeros in the strip $0 \leq \mathrm{Re}(s) \leq 1$ and no zeros outside that strip. It can be written as

$$\xi(s) = e^{Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}, \qquad (*)$$

where ρ runs through the zeros of $\xi(s)$ counted according to their multiplicities and

$$B = 1 + \frac{\gamma}{2} - \log(2\sqrt{\pi}).$$

Proof:

- By the previous lemma, we know that $\xi(s)$ has order 1.
- Noting that $\xi(0) = 1$, and using Hadamard's theorem we obtain (*) for some B.

Zeros of function ξ

- If $\xi(s)$ had only a finite number of zeros, (*) would imply the estimate $\log|\xi(s)| = O(|s|)$, which contradicts the fact that $\limsup_{|s|\to\infty} \frac{\log|\xi(s)|}{|s|\log|s|} = \frac{1}{2}$.
- For $\operatorname{Re} s > 1$, the zeta function $\zeta(s)$, and, consequently, $\xi(s)$, have no zeros in this range. By using the equation $\xi(s) = \xi(1-s)$ it follows that $\xi(s) \neq 0$ for $\operatorname{Re} s < 0$. Since $1 = \xi(0) = \xi(1) \neq 0$, the zeros of $\xi(s)$ lie in the strip $0 \leq \operatorname{Re}(s) \leq 1$.
- Show that $B = 1 + \frac{\gamma}{2} - \log(2\sqrt{\pi})$. □

Zeros of function ζ

- The zeta function $\zeta(s)$ has simple zeros at $s = -2, -4, -6, -8, \ldots$. These zeros are called the **trivial zeros**.
- For $\operatorname{Re} s > 1$, the zeta function $\zeta(s)$ has no zeros.
- In addition to the trivial zeros, the zeta function has infinitely many **nontrivial zeros** lying in the critical strip $0 \leq \operatorname{Re} s \leq 1$.
- In the critical strip $0 \leq \operatorname{Re} s \leq 1$ the zeros of

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

  are precisely the zeros of $\zeta(s)$.
- We also know that $\zeta(\sigma) \neq 0$ whenever $0 < \sigma < 1$.
- The nontrivial zeros of the zeta function are distributed symmetrically with respect to the lines $\operatorname{Re} s = 1/2$ and $\operatorname{Im} s = 0$, which follows from the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$