Lecture 9

Analytic branch of the complex logarithm

MATH 503, FALL 2025

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Complex logarithms and arguments

- The exponential function $S_\alpha \ni z \mapsto e^z \in \mathbb{C} \setminus \{0\}$ when restricted to the strip $S_\alpha = \{x + iy : \alpha \le y < \alpha + 2\pi\}$ is a one-to-one analytic map of this strip onto $\mathbb{C} \setminus \{0\}$ the nonzero complex numbers.

Definition

- We take $\log_\alpha$ to be the **inverse** of the exponential function restricted to the strip $S_\alpha = \{x + iy : \alpha \le y < \alpha + 2\pi\}$.
- We define $\arg_\alpha$ to the **imaginary part** of $\log_\alpha$.
- Consequently, $\log_\alpha(\exp z) = z$ for each $z \in S_\alpha$, and $\exp(\log_\alpha z) = z$ for all $z \in \mathbb{C}\setminus\{0\}$.

Definition

- The **principal branches** of the logarithm and argument functions, to be denoted by Log and Arg, are obtained by taking $\alpha = -\pi$.
- Thus, $\mathrm{Log} = \log_{-\pi}$ and $\mathrm{Arg} = \arg_{-\pi}$.

Complex logarithms and arguments

Theorem

(a) *If $z \neq 0$, then $\log_\alpha(z) = \log|z| + i\arg_\alpha(z)$, and $\arg_\alpha(z)$ is the unique number in $[\alpha, \alpha + 2\pi)$ such that*

$$z/|z| = e^{i\arg_\alpha(z)}.$$

In other words, the unique argument of z in $[\alpha, \alpha + 2\pi)$.

(b) *Let $R_\alpha = \{re^{i\alpha} : r \geq 0\}$. The functions $\log_\alpha$ and $\arg_\alpha$ are continuous at each point of the "slit" complex plane $\mathbb{C} \setminus R_\alpha$, and discontinuous at each point of R_α.*

(c) *The function $\log_\alpha$ is analytic on $\mathbb{C} \setminus R_\alpha$, and its derivative is given by $\log_\alpha'(z) = 1/z$.*

Complex logarithms and arguments

Proof:

(a) If $w = \log_\alpha(z)$ with $z \neq 0$, then $e^w = z$, hence

$$|z| = e^{\operatorname{Re} w}, \quad \text{and} \quad z/|z| = e^{i \operatorname{Im} w}.$$

Thus $\operatorname{Re} w = \log |z|$, and $\operatorname{Im} w$ is an argument of $z/|z|$. Since $\operatorname{Im} w$ is restricted to $[\alpha, \alpha + 2\pi)$ by definition of $\log_\alpha$, it follows that $\operatorname{Im} w$ is the unique argument for z that lies in the interval $[\alpha, \alpha + 2\pi)$.

(b) By (a), it suffices to consider $\arg_\alpha$. If $z_0 \in \mathbb{C} \backslash R_\alpha$ and $(z_n)_{n \in \mathbb{N}}$ is a sequence converging to z_0, then $\arg_\alpha(z_n)$ must converge to $\arg_\alpha(z_0)$. On the other hand, if $z_0 \in R_\alpha \backslash \{0\}$, there is a sequence $(z_n)_{n \in \mathbb{N}}$ converging to z_0 so that

$$\lim_{n \to \infty} \arg_\alpha(z_n) = \alpha + 2\pi \neq \arg_\alpha(z_0) = \alpha.$$

Continuous logarithms and arguments

Recall from Lecture 2 the following theorem:

Theorem

Let g be analytic on the open set Ω_1, and let f be a continuous complex-valued function on the open set Ω. Assume that

(i) $f(\Omega) \subseteq \Omega_1$,

(ii) g' *is never 0* ,

(iii) $g(f(z)) = z$ *for all* $z \in \Omega$ *(thus f is one-to-one).*

Then f is analytic on Ω and $f' = 1/(g' \circ f)$.

(c) By this theorem with $g = \exp$, $\Omega_1 = \mathbb{C}$, $f = \log_\alpha$, and $\Omega = \mathbb{C} \backslash R_\alpha$ and the fact that exp is its own derivative we obtain that

$$(\log_\alpha z)' = \frac{1}{z}.$$

This completes the proof of the theorem. □

Analytic logarithm

Definition

Let f be analytic on $\Omega \subseteq \mathbb{C}$. We say that g is an analytic logarithm of f if g is analytic on Ω and $e^{g(z)} = f(z)$ for $z \in \Omega$.

Remarks

Some conditions on Ω must be imposed:

- We know that $x \mapsto e^z$ is not injective in $\mathbb{C}$. To overcome this non-injectivity, we need to work with a branch of the logarithm.
- If $f(z) = 0$ for some $z \in \Omega$, then $0 = e^{g(z)} \neq 0$. Thus f must be nonvanishing on Ω.
- Let $A = \{z \in \mathbb{C} : 1/2 < |z| < 2\}$, so A is an open subset of $\mathbb{C}$, but there is no $g \in H(A)$ such that $\exp(g(z)) = z$ for $z \in A$. Otherwise, by differentiating $\exp(g(z)) = z$ we have $g'(z)\exp(g(z)) = 1$, which implies $g'(z) = 1/z$, thus $g(z)$ is a primitive of $1/z$ in A. But $1/z$ does not have a primitive in A. Thus Ω cannot be a region with holes.

Analytic logarithm

Theorem

Suppose that $\Omega \subseteq \mathbb{C}$ is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_\Omega(z)$ so that

(i) *F is holomorphic in Ω,*

(ii) *$e^{F(z)} = z$ for all $z \in \Omega$,*

(iii) *$F(r) = \log r$ whenever r is a real number and near 1.*

In other words, each branch $\log_\Omega(z)$ is an extension of the standard logarithm defined for positive numbers.

Proof: We shall construct F as a primitive of the function $1/z$. Since $0 \notin \Omega$, the function $f(z) = 1/z$ is holomorphic in Ω.

- We define
$$\log_\Omega(z) = F(z) = \int_\gamma f(w)dw,$$
where γ is any path in Ω connecting 1 to z.

Analytic logarithm

- Since Ω is simply connected, this definition does not depend on the path chosen. Indeed, if γ_0 is any other path in Ω joining 1 to z, then
$$\int_{\gamma_0} f(w)dw = \int_{\gamma_0-\gamma} f(w)dw + \int_{\gamma} f(w)dw = F(z),$$
since by the global Cauchy theorem $\int_{\gamma_0-\gamma} f(w)dw = 0$.
- Now we show that F is holomorphic and $F'(z) = 1/z$ for all $z \in \Omega$.
- Fix $z_0 \in \Omega$ and take $r > 0$ such that $D(z_0, r) \subseteq \Omega$, then by the global Cauchy theorem we have
$$F(z) - F(z_0) = \int_{[z,z_0]} f(w)dw.$$
- This implies that
$$F'(z_0) = \lim_{z\to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0)$$
as desired. This proves (i).

Analytic logarithm

- To prove (ii), it suffices to show that $ze^{-F(z)} = 1$. Using that $F'(z) = 1/z$, we differentiate the left-hand side, obtaining

$$\frac{d}{dz}\left(ze^{-F(z)}\right) = e^{-F(z)} - zF'(z)e^{-F(z)} = \left(1 - zF'(z)\right)e^{-F(z)} = 0.$$

- Since Ω is connected we conclude that $ze^{-F(z)}$ is constant.
- Evaluating this expression at $z = 1$, and noting that $F(1) = 0$, we find that this constant must be 1.
- Finally, if r is real and close to 1 we can choose as a path from 1 to r a line segment on the real axis so that

$$F(r) = \int_1^r \frac{dx}{x} = \log r$$

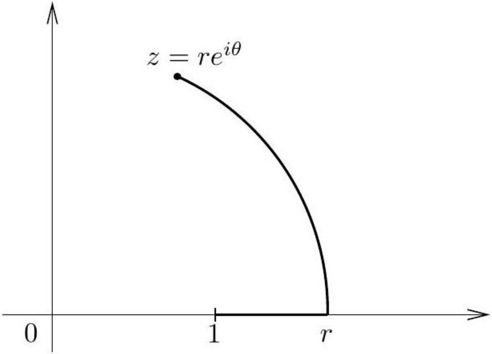
by the usual formula for the standard logarithm. This completes the proof of the theorem. □

Analytic logarithm

- For example, in the slit plane $\Omega = \mathbb{C} \setminus (-\infty, 0]$ we have the principal branch of the logarithm

$$\log z = \log r + i\theta$$

where $z = re^{i\theta}$ with $|\theta| < \pi$. (Here we drop the subscript Ω, and write simply $\log z$.) To prove this, we use the path of integration γ like here



Analytic logarithm

- If $z = re^{i\theta}$ with $|\theta| < \pi$, the path consists of the line segment from 1 to r and the arc η from r to z. Then

$$\begin{aligned} \log z &= \int_1^r \frac{dx}{x} + \int_\eta \frac{dw}{w} \\ &= \log r + \int_0^\theta \frac{ire^{it}}{re^{it}} dt \\ &= \log r + i\theta \end{aligned}$$

- An important observation is that in general

$$\log (z_1 z_2) \neq \log z_1 + \log z_2.$$

- For example, if $z_1 = e^{2\pi i/3} = z_2$, then for the principal branch of the logarithm, we have $\log z_1 = \log z_2 = 2\pi i/3$, and since $z_1 z_2 = e^{-2\pi i/3}$ we have

$$-\frac{2\pi i}{3} = \log (z_1 z_2) \neq \log z_1 + \log z_2.$$

Analytic logarithm

- Finally, for the principal branch of the logarithm the following Taylor expansion holds:

$$\begin{aligned}\log(1+z) &= z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n,\end{aligned}$$

  whenever $|z| < 1$.
- Indeed, the derivative of both sides equals $1/(1+z)$, so that they differ by a constant.
- Since both sides are equal to 0 at $z = 0$ this constant must be 0, and we have proved the desired Taylor expansion.

Analytic logarithm

- Having defined a logarithm on a simply connected domain, we can now define the powers z^α for any $\alpha \in \mathbb{C}$.
- If Ω is simply connected with $1 \in \Omega$ and $0 \notin \Omega$, we choose the branch of the logarithm with $\log 1 = 0$ as above, and define

$$z^\alpha = e^{\alpha \log z}.$$

- Note that $1^\alpha = 1$, and that if $\alpha = 1/n$, then

$$\begin{aligned} \left(z^{1/n}\right)^n &= \prod_{k=1}^{n} e^{\frac{1}{n} \log z} \\ &= e^{\sum_{k=1}^{n} \frac{1}{n} \log z} \\ &= e^{\log z} = z. \end{aligned}$$

Analytic logarithm

Theorem

If f is a nowhere vanishing holomorphic function in a simply connected region $\Omega \subseteq \mathbb{C}$, then there exists a holomorphic function g on Ω such that

$$f(z) = e^{g(z)}.$$

The function $g(z)$ in the theorem can be denoted by $\log f(z)$, and determines a "branch" of that logarithm.

Proof: Fix a point z_0 in Ω, and define a function

$$g(z) = \int_\gamma \frac{f'(w)}{f(w)} dw + c_0,$$

where γ is any path in Ω connecting z_0 to z, and c_0 is a complex number so that $e^{c_0} = f(z_0)$.

Analytic logarithm

- This definition is independent of the path γ since Ω is simply connected.
- Since f is a nowhere vanishing holomorphic function in Ω, thus f'/f is a holomorphic function in Ω, and by the global Cauchy theorem its primitive g exists in Ω and

$$g'(z) = \frac{f'(z)}{f(z)}.$$

- A simple calculation gives

$$\frac{d}{dz}\left(f(z)e^{-g(z)}\right) = 0$$

so that $f(z)e^{-g(z)}$ is constant.
- Evaluating this expression at z_0 we find $f\left(z_0\right)e^{-c_0} = 1$, so that $f(z) = e^{g(z)}$ for all $z \in \Omega$, and the proof is complete. □

Remarks on the global Cauchy theorem

Theorem

Let $\Omega \subseteq \mathbb{C}$ be a region. Then the following statements are equivalent:

(A) *For any closed path γ in Ω we have $\operatorname{Ind}_\gamma(z) = 0$ for any $z \in \mathbb{C} \setminus \Omega$.*

(B) *For any closed path γ in Ω we have*

$$\int_\gamma f(w)dw = 0 \quad \text{for every} \quad f \in H(\Omega).$$

(C) *For every $f \in H(\Omega)$ there exists $F \in H(\Omega)$ such that $F' = f$.*

(D) *For every $f \in H(\Omega)$ satisfying $1/f \in H(\Omega)$ there exists $g \in H(\Omega)$ such that $f = e^g$.*

(E) *For every $f \in H(\Omega)$ satisfying $1/f \in H(\Omega)$ there exists $g \in H(\Omega)$ such that $f = g^2$.*

Remarks on the global Cauchy theorem

Proof: We will show the following implications

$$(A) \implies (B) \implies (C) \implies (D) \implies (E) \implies (D) \implies (A).$$

Proof (A) $\implies$ (B): Let γ be a closed path in Ω such that

$$\mathrm{Ind}_\gamma(z) = 0 \quad \text{for any} \quad z \in \mathbb{C} \setminus \Omega.$$

- Pick $a \in \Omega \setminus \gamma^*$ and define

$$F(z) = (z-a)f(z).$$

- Using the global Cauchy theorem, we obtain

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \frac{1}{2\pi i}\int_\gamma \frac{F(z)}{z-a}dz = F(a)\cdot \mathrm{Ind}_\gamma(a) = 0,$$

since $F(a) = 0$. □

Remarks on the global Cauchy theorem

Proof (B) $\implies$ (C): Assume that for any closed path γ in Ω we have

$$\int_\gamma f(w)dw = 0 \quad \text{for every} \quad f \in H(\Omega).$$

- Fix $z_0 \in \Omega$ and $f \in H(\Omega)$. Define

$$F(z) = \int_\gamma f(w)dw,$$

where γ is any path in Ω connecting z_0 to z.

- This definition does not depend on the path chosen. Indeed, if γ_0 is any other path in Ω joining z_0 to z, then

$$\int_{\gamma_0} f(w)dw = \int_{\gamma_0-\gamma} f(w)dw + \int_\gamma f(w)dw = F(z),$$

since $\gamma_0 - \gamma$ is a closed path in Ω and by (B) we consequently have $\int_{\gamma_0-\gamma} f(w)dw = 0$.

Remarks on the global Cauchy theorem

- Now we show that F is holomorphic and $F' = f$ in Ω.
- Fix $a \in \Omega$ and take $r > 0$ such that $D(a, r) \subseteq \Omega$. By (B) we obtain

$$F(z) - F(a) = \int_{[z,a]} f(w)dw.$$

- Fix $\varepsilon > 0$ and observe that there exists $\delta \in (0, r)$ such that

$$|f(w) - f(a)| < \varepsilon \quad \text{whenever} \quad |w - a| < \delta.$$

- This implies that

$$\left| \frac{F(z) - F(a)}{z - a} - f(a) \right| < \varepsilon \quad \text{whenever} \quad |z - a| < \delta,$$

which proves

$$F'(a) = \lim_{z \to a} \frac{F(z) - F(a)}{z - a} = f(a).$$

as desired. □

Remarks on the global Cauchy theorem

Proof (C) $\Longrightarrow$ (D): Assume that for every $f \in H(\Omega)$ there exists $F \in H(\Omega)$ such that $F' = f$.

- Fix $f \in H(\Omega)$ satisfying $1/f \in H(\Omega)$. By (C) there exists $G \in H(\Omega)$ such that
$$G' = \frac{f'}{f}.$$
- Since $f \neq 0$ then $f(z_0) = e^{c_0}$ for some $c_0 \in \mathbb{C}$. Define
$$g(z) = G(z) - G(z_0) + c_0.$$
- Then $g' = G' = f'/f$ and consequently
$$(f \exp(-g))' = f' \exp(-g) - g' f \exp(-g) = 0.$$
- Thus $f \exp(-g)$ is constant in the region Ω, and in fact
$$f(z_0) \exp(-g(z_0)) = f(z_0) \exp(-c_0) = 1.$$
- Hence $f(z) = \exp(g(z))$ as desired. □

Remarks on the global Cauchy theorem

Proof (D) $\implies$ (E): Suppose that for every $f \in H(\Omega)$ satisfying $1/f \in H(\Omega)$ there exists $g \in H(\Omega)$ such that $f = e^g$.

- Note that
$$f = e^g = \left(e^{g/2}\right)^2$$
and we are done since $e^{g/2} \in H(\Omega)$. □

Proof (E) $\implies$ (D): Suppose that for every $f \in H(\Omega)$ satisfying $1/f \in H(\Omega)$ there exists $g \in H(\Omega)$ such that $f = g^2$.

- Then $f'/f \in H(\Omega)$. Note that
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \mathrm{Ind}_{f\circ\gamma}(0).$$
- By (E) it follows that for any $k \in \mathbb{N}$ there is $g_k \in H(\Omega)$ such that
$$g_k^{2^k} = f.$$

Remarks on the global Cauchy theorem

- Now observe that

$$\mathbb{Z} \ni \operatorname{Ind}_{f\circ\gamma}(0) = \frac{1}{2\pi i}\int_\gamma \frac{2^k g_k^{2^k-1}(z) g_k'(z)}{g_k^{2^k}(z)}dz = 2^k \operatorname{Ind}_{g_k\circ\gamma}(0) \in \mathbb{Z}.$$

- This 2^k divides $\operatorname{Ind}_{f\circ\gamma}(0)$ for every $k \in \mathbb{N}$, hence

$$0 = \operatorname{Ind}_{f\circ\gamma}(0) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz.$$

- Now, using implication from (B) to (C), we see that f'/f has a primitive in $H(\Omega)$, and we conclude by reasoning similarly to the implication from (C) to (D). □

Remarks on the global Cauchy theorem

Proof (D) $\implies$ (A): Suppose that for every $f \in H(\Omega)$ satisfying $1/f \in H(\Omega)$ there exists $g \in H(\Omega)$ such that $f = e^g$.

- Let $z \in \mathbb{C} \setminus \Omega$ and take $g \in H(\Omega)$ such that

$$w - z = \exp(g(w)) \quad \text{for} \quad w \in \Omega.$$

- Then we have that $1 = g'(w)(w - z)$. Hence

$$\frac{1}{w - z} = g'(w) \in H(\Omega).$$

  In other words, $\frac{1}{w-z}$ has a primitive in Ω.

- Thus

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z} dw = 0$$

  for any $z \in \mathbb{C} \setminus \Omega$ and any closed path γ in Ω. □