Analytic Number Theory
Lecture 11

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Vinogradov's mean value theorem

- Let $k, n \geqslant 1$ be integers, and $P \in \mathbb{Z}_+$ be a large integer. Let $J_{k,n}(P)$ be the number of integer solutions of the system of diophantine equations

$$\begin{aligned} x_1 + \cdots + x_k - x_{k+1} - \cdots - x_{2k} &= 0 \\ x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_{2k}^2 &= 0 \\ \vdots \qquad\qquad &\quad \vdots \\ x_1^n + \cdots + x_k^n - x_{k+1}^n - \cdots - x_{2k}^n &= 0, \end{aligned} \tag{HE}$$

  where $1 \leqslant x_1, \ldots, x_{2k} \leqslant P$.

Theorem (Vinogradov's mean value theorem)

Let $r \in \mathbb{N}$, $n \geq 2$, $k \geqslant n^2 + nr$, and $P \geqslant P_0$ and define

$$c_r = \frac{1}{2}\left(n^2 + n\right)\left(1 - \frac{1}{n}\right)^r.$$

Then

$$J_{k,n}(P) \leqslant (4n)^{4kr} P^{2k-(n^2+n)/2+c_r}. \tag{*}$$

Basic estimate

Lemma

For $N_1, N_2 \in \mathbb{Z}$ such that $N_1 < N_2$ and $\alpha \in \mathbb{R}_+$, we have

$$\Big|\sum_{n=N_1+1}^{N_2} e(\alpha n)\Big| \leqslant \min\{N_2 - N_1, (2\|\alpha\|)^{-1}\},$$

where $\|\alpha\|$ is the distance of α to the nearest integer, that is,

$$\|\alpha\| = \min\big\{\{\alpha\}, 1 - \{\alpha\}\big\} \leq 1/2.$$

Proof.

- If $\alpha \notin \mathbb{Z}$, then $\|\alpha\| > 0$ and $e(\alpha) \neq 0$. Thus

$$\Big|\sum_{n=N_1+1}^{N_2} e(\alpha n)\Big| = \Big|e(\alpha(N_1+1)) \sum_{n=0}^{N_2-N_1-1} e(\alpha n)\Big| = \left|\frac{e(\alpha(N_2-N_1)) - 1}{e(\alpha) - 1}\right|$$
$$\leq \frac{2}{|e(\alpha) - 1|} = \frac{1}{\sin(\pi\|\alpha\|)} \leq \frac{1}{2\|\alpha\|}.$$

- Otherwise we have that $\big|\sum_{n=N_1+1}^{N_2} e(\alpha n)\big| \leqslant N_2 - N_1$. □

Basic estimate

Lemma

Let $\alpha = a/q + \theta/q^2$, where $a, q \in \mathbb{Z}$ are integers such that $(a, q) = 1$, and $q \geqslant 1$, and $\theta \in [-1, 1]$. Then for arbitrary $\beta, U \in \mathbb{R}_+$ and $N \geqslant 1$ we have

$$\sum_{n \in [N]} \min \left\{ U, \frac{1}{\|\alpha n + \beta\|} \right\} \leqslant 6 \left(\frac{N}{q} + 1 \right) (U + q \log q).$$

Proof.

- Writing

$$\sum_{n \in [N]} = \sum_{n=1}^{q} + \sum_{n=q+1}^{2q} + \cdots$$

and changing indexes we obtain at most $N/q + 1$ sums of the form

$$S = \sum_{n \in [q]} \min \left\{ U, \|\alpha n + \beta_1\|^{-1} \right\},$$

with perhaps different β_1 's.

Proof

- It will be sufficient to prove
$$S \leqslant 6(U + q\log q). \tag{*}$$
- For $n \in [q]$ we have
$$\alpha n + \beta_1 = (an + \lfloor q\beta_1 \rfloor)\, q^{-1} + \theta'(n) q^{-2},$$
$$\theta'(n) = \theta n + (q\beta_1 - \lfloor q\beta_1 \rfloor)\, q, \quad |\theta'(n)| < 2q.$$
- We make the change of variable $m = an + [q\beta_1]$, so that by periodicity of $\|x\|$ and $(a, q) = 1$ we may assume that m runs over a complete system of residues $\pmod q$, and $|m| \leqslant q/2$. Thus
$$S = \sum_{|m| \leqslant q/2} \min\left\{U, \left\| \frac{m}{q} + \frac{\theta''(m)}{q} \right\|^{-1}\right\}, \quad \text{where} \quad |\theta''(m)| = |\theta'(m)/q| < 2.$$
- For $m = 0, \pm 1, \pm 2$, we estimate the min by U in the sum above, and for $2 < |m| \leqslant q/2$ we have
$$\left\| \frac{m}{q} + \frac{\theta''(m)}{q} \right\| > \frac{|m| - 2}{q},$$
hence
$$S \leqslant 5U + \sum_{2 < |m| \leqslant q/2} \frac{q}{|m| - 2} < 6(U + q\log q). \qquad \square$$

Exponential sums estimates

Lemma

Suppose that $t \geq 2$ *and* $N \in \mathbb{Z}_+$ *are large, and* $1 \leq M \leq N \leq t$*. Set* $r := \left\lfloor \frac{5.01 \log t}{\log N} \right\rfloor$*. Then*

$$\sum_{N<n\leq N+M} n^{-it} = O\Big(MN^{-4/5} \max_{N\leq n\leq 2N} |U(n)| + N^{4/5} + Mt^{-1/500}\Big),$$

where

$$U(n) := \sum_{x\in[K]} \sum_{y\in[K]} e\Big(\sum_{m\in[r]} \alpha_m x^m y^m\Big), \quad \alpha_m := \frac{(-1)^m t}{2\pi m n^m}, \quad K := N^{2/5}.$$

Proof.

- If $x, y \in [K]$, then

$$\begin{aligned}\sum_{N<n\leq N+M} n^{-it} &= \sum_{N-xy<n\leq N+M-xy} (n+xy)^{-it} \\ &= \sum_{N<n\leq N+M} (n+xy)^{-it} + \sum_{N-xy<n\leq N} (n+xy)^{-it} \\ &\quad - \sum_{N+M-xy<n\leq N+M} (n+xy)^{-it}.\end{aligned}$$

Proof

- Therefore, by averaging over $x, y \in [K]$ and shifting $n \to n + xy$, we obtain

$$\sum_{N<n\leq N+M} n^{-it} = K^{-2} \sum_{x\in[K]} \sum_{y\in[K]} \sum_{N<n\leq N+M} n^{-it}$$
$$= K^{-2} \sum_{x\in[K]} \sum_{y\in[K]} \sum_{N<n+xy\leq N+M} (n+xy)^{-it}$$
$$= K^{-2} \sum_{x\in[K]} \sum_{y\in[K]} \sum_{N<n\leq N+M} (n+xy)^{-it} + O(K^2),$$

since

$$\left| \sum_{N-xy<n\leq N} (n+xy)^{-it} \right| = \left| \sum_{N+M-xy<n\leq N+M} (n+xy)^{-it} \right| = O(K^2).$$

- Next, since $xy/n \leq N^{-1/5}$, observe that

$$\log\left(1 + \frac{xy}{n}\right) = \sum_{j=1}^{r} \frac{(-1)^{j-1}}{j} \left(\frac{xy}{n}\right)^j + O\left(\left(\frac{xy}{n}\right)^{5.01(\log t)/\log N}\right)$$
$$= \sum_{j=1}^{r} \frac{(-1)^{j-1}}{j} \left(\frac{xy}{n}\right)^j + O\left(t^{-(1+1/500)}\right).$$

Proof

- Observe that $n^{-it} = e(-t \log n/(2\pi))$. This implies that

$$\sum_{x\in[K]}\sum_{y\in[K]}(n+xy)^{-it} = \sum_{x\in[K]}\sum_{y\in[K]} e(-t\log(n+xy)/(2\pi))$$

$$= \sum_{x\in[K]}\sum_{y\in[K]} e\Big(\sum_{m\in[r]}\alpha_m x^m y^m\Big) e\Big(O\big(t^{-(1+1/500)}\big)\Big)$$

$$\sum_{x\in[K]}\sum_{y\in[K]} e\Big(\sum_{m\in[r]}\alpha_m x^m y^m\Big) + O\big(N^{4/5}t^{-(1+1/500)}\big),$$

where $\alpha_m := \frac{(-1)^m t}{2\pi m n^m}$ for $m \in [r]$. The conclusion of the lemma follows immediately. □

Further estimates

Lemma

Under the assumptions of the previous lemma, for any $k \in \mathbb{Z}_+$, we have

$$|U(n)| \leq N^{4/5}\left(\frac{J_{k,r}(N^{2/5})^2}{N^{8k/5}}\prod_{j=1}^{r}\sum_{-kN^{2j/5}\leq\mu_j\leq kN^{2j/5}}\min\left\{3kN^{2j/5},\frac{1}{\|\alpha_j\mu_j\|}\right\}\right)^{1/(4k^2)},$$

where $J_{k,r}\left(N^{2/5}\right)$ denotes the number of solutions $(x_1,\ldots,x_{2k}) \in \mathbb{Z}^{2k}$ of the simultaneous diophantine equations

$$\sum_{i=1}^{k} x_i^j = \sum_{i=k+1}^{2k} x_i^j, \quad \text{for all} \quad j \in [r] \quad \text{with} \quad 1 \leq x_i \leq N^{2/5}.$$

Proof.

- As before, let $K = N^{2/5}$ and by Höler's inequality we obtain

$$|U(n)|^{2k} \leq K^{2k-1}\sum_{x\in[K]}\left|\sum_{y\in[K]} e\Big(\sum_{m\in[r]}\alpha_m x^m y^m\Big)\right|^{2k}$$

$$= K^{2k-1}\sum_{x\in[K]}\sum_{y_1,\ldots,y_{2k}\in[K]} e\Big(\sum_{m\in[r]}\alpha_m x^m\Big(\sum_{i=1}^{k} y_i^m - \sum_{i=k+1}^{2k} y_i^m\Big)\Big).$$

Proof

- So if we let $J_{k,r}(K;\lambda_1,\dots,\lambda_r)$ denote the number of solutions $(y_1,\dots,y_{2k})\in\mathbb{Z}^{2k}$ of the simultaneous equations

$$\sum_{i=1}^{k} y_i^j = \sum_{i=k+1}^{2k} y_i^j + \lambda_j \quad \text{for all} \quad j\in[r],$$

with $1\le y_i\le K$, then we have

$$|U(n)|^{2k} \le K^{2k-1} \sum_{x\in[K]} \sum_{-kK\le\lambda_1\le kK} \cdots \sum_{-kK^r\le\lambda_r\le kK^r} J_{k,r}(K;\lambda_1,\dots,\lambda_r) e\Big(\sum_{m\in[r]} \alpha_m\lambda_m x^m\Big)$$
$$\le K^{2k-1} \sum_{-kK\le\lambda_1\le kK} \cdots \sum_{-kK^r\le\lambda_r\le kK^r} J_{k,r}(K;\lambda_1,\dots,\lambda_r)\Big|\sum_{x\in[K]} e\Big(\sum_{m\in[r]} \alpha_m\lambda_m x^m\Big)\Big|.$$

- We will abbreviate $\sum_{-kK^j\le\lambda_j\le kK^j}$ to $\sum_{\lambda_j}$. By Hölder's inequality we have

$$|U(n)|^{(2k)^2} \le K^{2k(2k-1)} \Bigg(\sum_{\lambda_1,\dots,\lambda_r} J_{k,r}(K;\lambda_1,\dots,\lambda_r)\Big|\sum_{x\in[K]} e\Big(\sum_{m\in[r]} \alpha_m\lambda_m x^m\Big)\Big|\Bigg)^{2k}$$
$$\le K^{2k(2k-1)} \Bigg(\sum_{\lambda_1,\dots,\lambda_r} J_{k,r}(K;\lambda_1,\dots,\lambda_r)^{\frac{2k}{2k-1}}\Bigg)^{2k-1} \sum_{\lambda_1,\dots,\lambda_r} \Big|\sum_{x\in[K]} e\Big(\sum_{m\in[r]} \alpha_m\lambda_m x^m\Big)\Big|^{2k}.$$

Proof

- Using $J_{k,r}(K;\lambda_1,\ldots,\lambda_r) \le J_{k,r}(K)$, we obtain

$$\left(\sum_{\lambda_1,\ldots,\lambda_r} J_{k,r}(K;\lambda_1,\ldots,\lambda_r)^{\frac{2k}{2k-1}}\right)^{2k-1} \le K^{2k(2k-1)} J_{k,r}(K),$$

since

$$\sum_{\lambda_1,\ldots,\lambda_r} J_{k,r}(K;\lambda_1,\ldots,\lambda_r) = K^{2k}.$$

- Therefore, we have

$$|U(n)|^{(2k)^2} \le K^{4k(2k-1)} J_{k,r}(K) \sum_{\lambda_1,\ldots,\lambda_r} \left| \sum_{x\in[K]} e\Big(\sum_{m\in[r]} \alpha_m \lambda_m x^m\Big)\right|^{2k}.$$

- Expanding the $2k$-th power of the last sum as before, we obtain that

$$\begin{aligned} &|U(n)|^{(2k)^2} \\ &\le K^{4k(2k-1)} J_{k,r}(K) \sum_{\lambda_1,\mu_1,\ldots,\lambda_r,\mu_r} J_{k,r}(K;\mu_1,\ldots,\mu_r)\, e\Big(\sum_{m\in[r]} \alpha_m \mu_m \lambda_m\Big) \\ &\le K^{4k(2k-1)} \left(J_{k,r}(K)\right)^2 \sum_{\mu_1,\ldots,\mu_r} \prod_{m\in[r]} \left| \sum_{-kZ^m \le \lambda_m \le kZ^m} e\left(\alpha_m \mu_m \lambda_m\right)\right|. \end{aligned}$$

Zeta sum estimate of Vinogradov and Korobov

- We immediately deduce that

$$\begin{aligned}&|U(n)|^{(2k)^2}\\ &\le K^{4k(2k-1)}\left(J_{k,r}(K)\right)^2 \sum_{|\mu_1|\le kK}\cdots\sum_{|\mu_r|\le kK^r}\prod_{m\in[r]}\min\left\{3kK^m, \frac{1}{\|\alpha_m\mu_m\|}\right\}.\end{aligned}$$

- Raising both sides to the power $1/\left(4k^2\right)$, and remembering that $K = N^{2/5}$, the bound claimed in the lemma follows. □

Theorem (Vinogradov and Korobov)

There exists a small absolute constant $c > 0$ such that the following is true. For any $1 \le M \le N \le t$, one has

$$\begin{aligned}\left|\sum_{N<n\le N+M} n^{-it}\right| &= O\left(Me^{-c\left(\log^3 N\right)/\log^2(t+2)} + N^{4/5}\right)\\ &= O\left(Ne^{-c\left(\log^3 N\right)/\log^2(t+2)}\right).\end{aligned}$$

Proof

- We may assume that $t \geq 2$ is large and that $N \geq e^{\log^{2/3} t}$, since otherwise the theorem is trivial by adjusting the O constant appropriately.
- Setting $r = \left\lfloor \frac{5.01 \log t}{\log N} \right\rfloor$, we have

$$\left| \sum_{N<n\leq N+M} n^{-it} \right| = O\Big(MN^{-4/5} \max_{N\leq n\leq 2N} |U(n)| + N^{4/5} + Mt^{-1/500} \Big),$$

where

$$U(n) = \sum_{x\in[N^{2/5}]} \sum_{y\in[N^{2/5}]} e\left(\alpha_1 xy + \ldots + \alpha_r x^r y^r\right), \quad \alpha_j = \frac{(-1)^j t}{2\pi j n^j}.$$

- Since $t \geq N$ we have $t^{-1/500} = e^{-(1/500)\log t} \leq e^{-(1/500)(\log^3 N)/\log^2 t}$, so the last two terms satisfy the desired bound.
- By the previous lemma, for any $N \leq n \leq 2N$ and any $k \in \mathbb{N}$ we have

$$\frac{|U(n)|}{N^{4/5}} \leq \left(\frac{J_{k,r}(N^{2/5})^2}{N^{8k/5}} \prod_{j\in[r]} \sum_{|\mu_j|\leq kN^{2j/5}} \min\left\{ 3kN^{2j/5}, \frac{1}{\|\alpha_j \mu_j\|} \right\} \right)^{1/(4k^2)}.$$

Proof

- We take $k = Cr^2 \in \mathbb{Z}_+$, where $C \geq 1$ is a constant that we will choose later. Since $k \geq r^2$, Vinogradov's mean value theorem implies that

$$J_{k,r}(N^{2/5})^2 \leq (4r)^{8kF} N^{(4/5)(2k-(1-\delta)(1/2)r(r+1))},$$

  where $F = \lfloor (C-1)r \rfloor$, and $\delta = (1-1/r)^F$.
- Taking $C(r,k,N) := (4r)^{8Ckr} N^{-(4/5)(1-\delta)r(r+1)/2}$, we see

$$\frac{|U(n)|}{N^{4/5}} \leq \left(C(r,k,N) \prod_{j \in [r]} \sum_{|\mu_j| \leq kN^{2j/5}} \min \left\{ 3kN^{2j/5}, \frac{1}{\|\alpha_j \mu_j\|} \right\} \right)^{1/(4k^2)}.$$

- It remains to bound the sums over μ_j. We always have the trivial bound $O(k^2 N^{4j/5})$, and if $\alpha_j = a_j/q_j + \theta_j/q_j^2$ for some $a_j \in \mathbb{Z}$, $q_j \in \mathbb{Z}_+$, such that $(a_j, q_j) = 1$ and $|\theta_j| \leq 1$, then we have

$$\sum_{|\mu_j| \leq kN^{2j/5}} \min \left\{ 3kN^{2j/5}, \frac{1}{\|\alpha_j \mu_j\|} \right\} = O\left(\left(\frac{kN^{2j/5}}{q_j} + 1 \right) \left(kN^{2j/5} + q_j \log q_j \right) \right)$$

$$= O\left(\max \left\{ \frac{k^2 N^{4j/5}}{q_j}, q_j \right\} \log(q_j + 1) \right).$$

Proof

- But if $j \geq (\log t)/\log N$ then, remembering that $N \leq n \leq 2N$, we have

$$\alpha_j = \frac{(-1)^j t}{2\pi j n^j} = \frac{(-1)^j}{q_j} + \frac{\theta_j}{q_j^2}, \quad \text{and} \quad \theta_j = \frac{(-1)^{j+1}\left\{2\pi j n^j/t\right\} q_j}{2\pi j n^j/t},$$

where $q_j := \lfloor 2\pi j n^j/t \rfloor \geq 1$ and $|\theta_j| \leq 1$.

- If $2(\log t)/\log N \leq j \leq 3(\log t)/\log N$ then $q_j \geq n^{j-(\log t)/\log N} \geq N^{j/2}$ and also $q_j = O(jn^j N^{-j/3}) = O(j2^j N^{2j/3})$, so we certainly have

$$\sum_{|\mu_j| \leq kN^{2j/5}} \min\left\{3kN^{2j/5}, \frac{1}{\|\alpha_j \mu_j\|}\right\} = O\left(\frac{k^2 N^{4j/5}}{N^{j/10}}\right),$$

whenever $2(\log t)/\log N \leq j \leq 3(\log t)/\log N$.

- Remembering that $C(r,k,N) := (4r)^{8Ckr} N^{-(4/5)(1-\delta)r(r+1)/2}$, we see

$$\frac{|U(n)|}{N^{4/5}} \leq \left(C(r,k,N) \prod_{j=1}^{r} \left(Dk^2 N^{4j/5}\right) \prod_{2(\log t)/\log N \leq j \leq 3(\log t)/\log N} \frac{1}{N^{j/10}}\right)^{\frac{1}{4k^2}}$$

$$\leq \left(C(r,k,N)\left(Dk^2\right)^r N^{(4/5)r(r+1)/2} N^{-(1/10)((\log t)/\log N)^2}\right)^{\frac{1}{4k^2}},$$

where D is a large absolute constant.

Proof

- Note that $C(r,k,N)N^{(4/5)r(r+1)/2} = (4r)^{8Ckr}N^{(4/5)\delta r(r+1)/2}$.
- Using $k = Cr^2$ and $r = \left\lfloor \frac{5.01 \log t}{\log N} \right\rfloor$, and choosing C large enough, we have $\delta \leq 1/280$, and therefore $(4/5)\delta r(r+1)/2 \leq 14\delta((\log t)/\log N)^2 \leq (1/20)((\log t)/\log N)^2$, and therefore

$$\frac{|U(n)|}{N^{4/5}} \leq \left((4r)^{8Ckr}\left(Dk^2\right)^r N^{-(1/20)((\log t)/\log N)^2}\right)^{\frac{1}{4k^2}}$$
$$= \left((4r)^{8C^2r^3}\left(DC^2r^4\right)^r N^{-(1/20)(\log t/\log N)^2}\right)^{1/\left(4C^2r^4\right)}$$

The dominant term inside the bracket is $N^{-(1/20)(\log t/\log N)^2}$, and so the conclusion of the theorem follows. □

Exercise

Let $\sigma_0 > 0$. We have uniformly for $x \geqslant 1$, $\sigma \geqslant \sigma_0$ and $|t| \leqslant \pi x$ the following formula

$$\zeta(s) = \sum_{n \in [x]} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + O\left(x^{-\sigma}\right).$$

Improved bounds on the Riemann zeta function

Theorem (Richert)

There exists a large absolute constant $C > 0$ such that the following is true. For any large $t \geq 2$ and any $0 < \sigma \leq 1$, we have

$$\zeta(\sigma + it) = O\Big(t^{C(1-\sigma)^{3/2}} \log^{2/3} t\Big).$$

In particular, if $\sigma \geq 1 - 1/\log^{2/3} t$ then $\zeta(\sigma + it) = O(\log^{2/3} t)$.

Proof.

- By the approximation formula, we have $\zeta(\sigma + it) = \sum_{n \in [t]} \frac{1}{n^s} + O(1)$.
- Suppose first that $1 - 1/\log^{2/3} t \leq \sigma \leq 1$. Then

$$\begin{aligned} |\zeta(\sigma + it)| &= \Big| \sum_{1 \leq n \leq e^{\log^{2/3} t}} \frac{1}{n^{\sigma+it}} \Big| + \Big| \sum_{e^{\log^{2/3}} < n \leq t} \frac{1}{n^{\sigma+it}} \Big| + O(1) \\ &\leq \sum_{1 \leq n \leq e^{\log^{2/3} t}} \frac{1}{n} + \sum_{\lfloor \log^{2/3} t \rfloor \leq j \leq \log t} \Big| \sum_{e^j < n \leq e^{j+1}} \frac{1}{n^{\sigma+it}} \Big| \\ &= O\Bigg(\log^{2/3} t + \sum_{\lfloor \log^{2/3} t \rfloor \leq j \leq \log t} \Big| \sum_{e^j < n \leq e^{j+1}} \frac{1}{n^{\sigma+it}} \Big| \Bigg), \end{aligned}$$

Proof

- The sequence $\frac{1}{n^\sigma}$ is monotone decreasing, hence, Abel's summation lemma implies that

$$\left|\sum_{e^j<n\le e^{j+1}} n^{-(\sigma+it)}\right| = O\left(e^{-j\sigma} \max_{e^j<n'\le e^{j+1}} \left|\sum_{e^j<n\le n'} n^{-it}\right|\right).$$

- By the previous theorem, we have

$$\max_{e^j<n'\le e^{j+1}} \left|\sum_{e^j<n\le n'} n^{-it}\right| = O\left(e^j e^{-cj^3/\log^2 t}\right).$$

- Thus we have

$$\begin{aligned} |\zeta(\sigma+it)| &= O\left(\log^{2/3} t + \sum_{\lfloor \log^{2/3} t\rfloor \le j\le \log t} e^{j(1-\sigma)-cj^3/\log^2 t}\right) \\ &= O\left(\log^{2/3} t + \sum_{\lfloor \log^{2/3} t\rfloor \le j\le \log t} e^{j/\log^{2/3} t - c(j/\log^{2/3} t)^3}\right) \\ &= O\left(\log^{2/3} t + \sum_{r=1}^{\lfloor \log^{1/3} t\rfloor} \sum_{j=r\lfloor \log^{2/3} t\rfloor}^{(r+1)\lfloor \log^{2/3} t\rfloor} e^{j/\log^{2/3} t - c(j/\log^{2/3} t)^3}\right). \end{aligned}$$

- The final sum is $O(\log^{2/3} t \sum_{r\ge 1} e^{r-cr^3})$, and this is clearly $O(\log^{2/3} t)$.

Proof

- If instead $\sigma < 1 - 1/\log^{2/3} t$ then one can proceed in a similar way, but breaking the sum over n at a different place to obtain a saving when summing over j. In fact, for any large constant C we obtain that

$$\begin{aligned}
\zeta(\sigma+it) &= \sum_{n \le e^{C\log t\sqrt{1-\sigma}}} \frac{1}{n^{\sigma+it}} + \sum_{e^{C\log t\sqrt{1-\sigma}} < n \le t} \frac{1}{n^{\sigma+it}} + O(1) \\
&\ll \frac{e^{C\log t(1-\sigma)^{3/2}}}{1-\sigma} + \sum_{\lfloor C\log t\sqrt{1-\sigma}\rfloor \le j \le \log t} \left| \sum_{e^j < n \le e^{j+1}} \frac{1}{n^{\sigma+it}} \right| \\
&\ll t^{C(1-\sigma)^{3/2}} \log^{2/3} t + \sum_{\lfloor C\log t\sqrt{1-\sigma}\rfloor \le j \le \log t} e^{j(1-\sigma) - cj^3/\log^2 t} \\
&\ll t^{C(1-\sigma)^{3/2}} \log^{2/3} t + \sum_{\lfloor C\log t\sqrt{1-\sigma}\rfloor \le j \le \log t} e^{j(1-\sigma)-cj(1-\sigma)C^2}.
\end{aligned}$$

Provided C is large enough the exponents in the sum over j will all be negative, and one can bound it as we did above by breaking into intervals of length $\lfloor C\log t\sqrt{1-\sigma}\rfloor$.

Theorem of Landau

- In fact, any order of magnitude of $\zeta(s)$ in a certain domain implies a zero-free region as may be seen in the next result devised by Landau.

Theorem (Landau)

Let $\theta(t)$ and $\phi(t)$ be positive functions such that $\theta(t)$ is decreasing, $\phi(t)$ is increasing and $e^{-\phi(t)} \leqslant \theta(t) \leqslant \frac{1}{2}$. Assume that $\zeta(s) = O(e^{\phi(t)})$ in the region $\sigma \geqslant 1 - \theta(t)$ and $t \geqslant 2$. Then the following assertions hold.

(i) *There exists an absolute constant $c_0 > 0$ such that $\zeta(s)$ has no zero in the region*

$$\sigma \geqslant 1 - c_0 \frac{\theta(2t+1)}{\phi(2t+1)}.$$

(ii) *In the region $\sigma \geqslant 1 - (c_0/2)\,\theta(2t+2)\phi(2t+2)^{-1}$, we have*

$$\frac{1}{\zeta(s)} = O\left(\frac{\theta(2t+2)}{\varphi(2t+2)}\right) \quad \text{and} \quad \frac{\zeta'(s)}{\zeta(s)} = O\left(\frac{\theta(2t+2)}{\varphi(2t+2)}\right).$$

Vinogradov–Korobov zero-free region

Theorem

There exists a small absolute constant $c > 0$ such that the zeta function has no zeros $s = \sigma + it$ it in the region

$$\left\{ s = \sigma + it \in \mathbb{C} : \sigma \geq 1 - c \left(\log^{2/3}(|t| + 2)(\log\log(|t| + 3))^{1/3} \right)^{-1} \right\}.$$

Proof.

- Richert's theorem tells us that, for all $t \geq t_0$ (a large constant), we have

$$\zeta(\sigma + it) = O\Big(t^{C(1-\sigma)^{3/2}} \log^{2/3} t \Big) = O\Big(e^{C(1-\sigma)^{3/2} \log t + (2/3) \log\log t} \Big).$$

- In particular, if $\sigma \geq 1 - ((\log\log t)/\log t)^{2/3}$ then we have

$$\zeta(\sigma + it) = O\big(e^{(C+2/3) \log\log t} \big),$$

  so we can apply Landau's theorem with the choices

$$\phi(t) = (C + 2/3) \log\log t \quad \text{and} \quad \theta(t) = ((\log\log t)/\log t)^{2/3}.$$

Proof

- Hence we conclude that $\zeta(\sigma + it) \neq 0$ in the region

$$\begin{aligned}\sigma &\geq 1 - c\frac{\theta(2t+1)}{\phi(2t+1)} \\ &= 1 - \frac{c}{(C+2/3)\log^{2/3}(2t+1)(\log\log(2t+1))^{1/3}}, \quad t \geq t_0\end{aligned}$$

- By relabelling the constants, this gives the assertion of the corollary when $t \geq t_0$. One can obtain the analogous result for $t < t_0$ using the known zero-free regions and using symmetry.

Theorem (PNT with the best error term to date)

There exists an absolute constant $c \in (0,1)$ such that as $x \to \infty$, one has

$$\begin{aligned}\psi(x) &= x + O\left(x\exp\left(-c(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right), \\ \pi(x) &= \mathrm{Li}(x) + O\left(x\exp\left(-c(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right).\end{aligned}$$

Weyl's method

- Consider a polynomial $\psi(x)$ with real coefficients and the exponential sum
$$T(X) = \sum_{1 \leqslant x \leqslant X} e(\psi(x)).$$
- An idea of Weyl permits one to reduce the degree inductively to obtain an estimate in terms of exponential sums with linear phases.
- Since $\overline{e(\psi(x))} = e(-\psi(x))$, one finds that
$$|T(X)|^2 = \sum_{1 \leqslant x,y \leqslant X} e(\psi(y) - \psi(x)) = \sum_{|h|<X} \sum_{\substack{1 \leqslant x \leqslant X \\ 1 \leqslant x+h \leqslant X}} e(\psi(x+h) - \psi(x)),$$
in which we have made the change of variable $y = x + h$. But one has $\psi(x+h) - \psi(x) = h\psi_1(x;h)$, where $\psi_1(x;h)$ is a polynomial in x and h having degree $\deg(\psi) - 1$ with respect to x. Thus, we have
$$|T(X)|^2 \leqslant \sum_{|h|<X} \left| \sum_{\substack{1 \leqslant x \leqslant X \\ 1 \leqslant x+h \leqslant X}} e\left(h\psi_1(x;h)\right) \right|$$
with the inner exponential sum being one having a polynomial argument of degree one less than that of the original sum $T(X)$.

Weyl's method

- This idea may now be used repeatedly to reduce the degree of the exponential sum under consideration until one is left in the linear situation amenable to exponential sums with linear phases.
- When $\psi(x)$ is a real-valued function of x, we denote by Δ_1 the difference operator defined by

$$\Delta_1(\psi(x); h) = \psi(x+h) - \psi(x).$$

- Then we define Δ_j by

$$\begin{aligned}\Delta_j(\psi(x); \mathbf{h}) &= \Delta_j\left(\psi(x); h_1, \ldots, h_j\right) \\ &= \Delta_1\left(\Delta_{j-1}\left(\psi(x); h_1, \ldots, h_{j-1}\right); h_j\right) \quad \text{for} \quad j \geqslant 2.\end{aligned}$$

- By convention, we take $\Delta_0(\psi(x); h) = \psi(x)$.
- One may verify that when $j \in [k]$, one has

$$\Delta_j\left(x^k; \mathbf{h}\right) = h_1 \ldots h_j p_j\left(x; h_1, \ldots, h_j\right),$$

  where p_j is a polynomial in x of degree $k-j$ with leading coefficient $k!/(k-j)!$.

Weyl's method

- By the linearity of the operator Δ_j, one sees that

$$\Delta_j\left(a_k x^k + \ldots + a_1 x; \mathbf{h}\right) = \sum_{i=1}^{k} a_i \Delta_j\left(x^i; \mathbf{h}\right)$$

  and so one is able easily to calculate $\Delta_j(p(x);\mathbf{h})$, for polynomial $p(x)$, by using what is known for monomials.
- For the special case $p(x) = x^k$, we have

$$\Delta_{k-1}(x^k; \mathbf{h}) = h_1 \cdots h_{k-1} k! \Big(x - \frac{h_1 + \ldots + h_{k-1}}{2}\Big).$$

- We note, in particular, that when $j > \deg(p)$, then one has

$$\Delta_j(p(x); \mathbf{h}) = 0.$$

Weyl differencing

Lemma (Weyl differencing)

Let $\psi(x)$ be a real-valued arithmetic function, and put

$$F(\psi) = \sum_{1 \leqslant x \leqslant X} e(\psi(x)).$$

Then for each $j \in \mathbb{Z}_+$, one has

$$|F(\psi)|^{2^j} \leqslant (2X)^{2^j - j - 1} \sum_{|h_1|<X} \cdots \sum_{|h_j|<X} \sum_{x \in I_j(\mathbf{h})} e\left(\Delta_j(\psi(x); \mathbf{h})\right), \qquad (*)$$

where $I_j(\mathbf{h})$ denotes the interval of integers defined by setting $I_0(h) = [1, X]$, and, when $j \in \mathbb{Z}_+$, by

$$I_j\left(h_1, \ldots, h_j\right) = I_{j-1}\left(h_1, \ldots, h_{j-1}\right) \cap \left\{x \in [1, X] : x + h_j \in I_{j-1}\left(h_1, \ldots, h_{j-1}\right)\right\}.$$

Note that

$$I_j\left(h_1, \ldots, h_j\right) \subseteq I_{j-1}\left(h_1, \ldots, h_{j-1}\right) \subseteq \ldots \subseteq I_1\left(h_1\right) \subseteq [1, X].$$

Proof

- We proceed by induction. When $j=1$, one has

$$
\begin{aligned}
|F(\psi)|^2 &= \sum_{1\leqslant x\leqslant X}\sum_{1\leqslant y\leqslant X} e(\psi(y)-\psi(x))\\
&= \sum_{1\leqslant x\leqslant X}\sum_{1-x\leqslant h_1\leqslant X-x} e\left(\psi\left(x+h_1\right)-\psi(x)\right)\\
&= \sum_{|h_1|<X}\sum_{x\in I_1(h_1)} e\left(\Delta_1\left(\psi(x);h_1\right)\right),
\end{aligned}
$$

  where $I_1\left(h_1\right)=[1,X]\cap[1-h_1,X-h_1]$ and (*) follows for $j=1$.
- Suppose now that (*) has been proved for all $j\in[J-1]$. Then, by the Cauchy–Schwarz inequality, one finds that

$$
|F(\psi)|^{2^J}=\left(|F(\psi)|^{2^{J-1}}\right)^2\leqslant\left((2X)^{2^{J-1}-J}\right)^2\left(\sum_{|h_1|<X}\cdots\sum_{|h_{J-1}|<X}1\right)\Xi(\psi),
$$

  where

$$
\Xi(\psi)=\sum_{|h_1|<X}\cdots\sum_{|h_{J-1}|<X}\left|\sum_{x\in I_{J-1}(\mathbf{h})} e\left(\Delta_{J-1}(\psi(x);\mathbf{h})\right)\right|^2.
$$

Proof

- As in the case $j = 1$, one opens the square to write

$$\Big|\sum_{x\in I_{J-1}(\mathbf{h})} e\left(\Delta_{J-1}(\psi(x);\mathbf{h})\right)\Big|^2 = \sum_{|h_J|<X}\sum_{x\in I_J(\mathbf{h})} e\left(\Delta_1\left(\Delta_{J-1}(\psi(x);\mathbf{h});h_J\right)\right),$$

where

$$I_J\left(\mathbf{h},h_J\right) = I_{J-1}(\mathbf{h}) \cap \{x\in[1,X] : x+h_J \in I_{J-1}(\mathbf{h})\}.$$

- We therefore conclude that

$$|F(\psi)|^{2^J} \leqslant (2X)^{2^J-J-1}\sum_{|h_1|<X}\cdots\sum_{|h_J|<X}\sum_{x\in I_J(\mathbf{h})} e\left(\Delta_J(\psi(x);\mathbf{h})\right).$$

- This gives (*) for $j = J$, and the conclusion follows by induction. □

Weyl differencing for polynomial phases

- If we apply Weyl differencing $k-1$ times to the exponential sum

$$f(\alpha) = \sum_{1 \leqslant x \leqslant X} e\left(\alpha x^k\right),$$

then we obtain a bound of the shape

$$|f(\alpha)|^{2^{k-1}} = O\left(X^{2^{k-1}-k} \sum_{\substack{h_1,\dots,h_{k-1} \\ |h_i|<X}} \sum_{x \in I(\mathbf{h})} e\left(k!\alpha h_1 \dots h_{k-1}\left(x + \frac{h_1+\dots+h_{k-1}}{2}\right)\right)\right).$$

- This gives us an exponential sum with a linear phase, but we will need to average over the products $h_1 \dots h_{k-1}$.
- This entails a discussion of the r-fold divisor function

$$\tau_r(n) = \sum_{\substack{d_i \in \mathbb{N} \\ d_1 \dots d_r = n}} 1 \quad \text{for} \quad r = k-1.$$

- Invoking bounds for the divisor function we have that $\tau_r(n) = O_{r,\varepsilon}(n^{\varepsilon})$ for each $r \in \mathbb{N}$ and $\varepsilon > 0$.

Weyl's inequality

Lemma (Weyl's inequality)

Let $k \geqslant 2$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|\alpha_k - a/q| \leqslant q^{-2}$. Then

$$\Big| \sum_{1 \leqslant x \leqslant X} e\left(\alpha_1 x + \ldots + \alpha_k x^k\right) \Big| = O\Big(X^{1+\varepsilon}\left(q^{-1} + X^{-1} + qX^{-k}\right)^{2^{1-k}}\Big).$$

Proof.

- Write $\psi(x) = \alpha_1 x + \ldots + \alpha_k x^k$ and

$$F(\boldsymbol{\alpha}) = \sum_{1 \leqslant x \leqslant X} e(\psi(x)).$$

- If $q > X^k$, the desired estimate is trivial from the bound

$$|F(\boldsymbol{\alpha})| \leqslant \sum_{1 \leqslant x \leqslant X} 1 \leqslant X.$$

- Thus, we may suppose without loss that $q \leqslant X^k$.

Proof

- By applying the Weyl differencing lemma with $j = k-1$, we obtain the bound
$$|F(\boldsymbol{\alpha})|^{2^{k-1}} = O\Bigg(X^{2^{k-1}-k} \sum_{|h_1|<X} \cdots \sum_{|h_{k-1}|<X} \Upsilon(\mathbf{h}) \Bigg),$$
where
$$\Upsilon(\mathbf{h}) = \sum_{x \in I_{k-1}(\mathbf{h})} e\left(\Delta_{k-1}(\psi(x); \mathbf{h})\right),$$
and $I_{k-1}(\mathbf{h})$ is a suitable interval of integers contained in $[1, X]$.
- We note that
$$\Delta_{k-1}(\psi(x); \mathbf{h}) = k! h_1 \dots h_{k-1} x \alpha_k + \gamma,$$
where $\gamma = \gamma(\boldsymbol{\alpha}; \mathbf{h})$ is independent of x.
- Thus, we obtain that
$$\Upsilon(\mathbf{h}) = O\Big(\min\left\{ X, \|k! h_1 \dots h_{k-1} \alpha_k\|^{-1} \right\} \Big),$$
whence
$$|F(\boldsymbol{\alpha})|^{2^{k-1}} = O\Bigg(X^{2^{k-1}-k} \sum_{|h_1|<X} \cdots \sum_{|h_{k-1}|<X} \min\left\{ X, \|k! h_1 \dots h_{k-1} \alpha_k\|^{-1} \right\} \Bigg).$$

Proof

- Separating the summands for which $h_1 \cdots h_{k-1} = 0$, we are led from here to the bound

$$|F(\boldsymbol{\alpha})|^{2^{k-1}} = O\left(X^{2^{k-1}-k}\left(X^{k-1} + \sum_{1 \leqslant n \leqslant k!X^{k-1}} \tau_k(n) \min\left\{X, \|n\alpha_k\|^{-1}\right\}\right)\right).$$

- Since $\tau_k(n) = O(X^{\varepsilon})$, recalling our assumption that $q \leqslant X^k$, and using the bound

$$\sum_{n \in [N]} \min\left\{U, \frac{1}{\|\alpha n + \beta\|}\right\} \leqslant 6\left(\frac{N}{q} + 1\right)(U + q \log q),$$

with $N = k!X^{k-1}$ and $U = X$, we therefore obtain the estimate

$$F(\boldsymbol{\alpha}) = O\left(X^{1-2^{1-k}} + X^{1+\varepsilon}\left(q^{-1} + X^{-1} + X^{1-k} + qX^{-k}\right)^{2^{1-k}}\right),$$

and the conclusion of the lemma follows. □

Hua's lemma

Lemma (Hua's lemma)

Let $X \in \mathbb{R}_+$ be a large real number and write

$$f(\alpha) = \sum_{1 \leqslant x \leqslant X} e\left(\alpha x^k\right).$$

Suppose that $k \geqslant 2$ and $j \in [k]$. Then one has

$$\int_0^1 |f(\alpha)|^{2^j} \, \mathrm{d}\alpha = O\big(X^{2^j - j + \varepsilon}\big).$$

Proof.

- We proceed by induction on $j \in [k]$. When $j = 1$, then by orthogonality

$$\int_0^1 |f(\alpha)|^2 \, \mathrm{d}\alpha = \int_0^1 f(\alpha) f(-\alpha) \mathrm{d}\alpha = \#\left\{1 \leqslant x, y \leqslant X : x^k = y^k\right\} = \lfloor X \rfloor,$$

  and the base case $j = 1$ of the induction is verified.
- Now suppose that the desired conclusion has been established already when $j \in [J]$ for some integer $J \in \mathbb{Z}_+$ satisfying $1 \leqslant J < k$.

Proof

- By the Weyl differencing lemma we see that
$$|f(\alpha)|^{2^J} \leqslant (2X)^{2^J-J-1} \sum_{|h_1|<X} \cdots \sum_{|h_J|<X} \sum_{x\in I_J(\mathbf{h})} e\left(\alpha \Delta_J\left(x^k;\mathbf{h}\right)\right),$$
where $I_J(\mathbf{h})$ is a suitable subinterval of $[1,X]$. It follows that
$$\begin{aligned}\int_0^1 |f(\alpha)|^{2^{J+1}}\,\mathrm{d}\alpha &= \int_0^1 f(\alpha)^{2^{J-1}} f(-\alpha)^{2^{J-1}} |f(\alpha)|^{2^J}\,\mathrm{d}\alpha \\ &\leqslant (2X)^{2^J-J-1}T, \qquad (*)\end{aligned}$$
where
$$T = \sum_{|h_1|<X} \cdots \sum_{|h_J|<X} \sum_{x\in I_J(\mathbf{h})} \int_0^1 f(\alpha)^{2^{J-1}} f(-\alpha)^{2^{J-1}} e\left(\alpha \Delta_J\left(x^k;\mathbf{h}\right)\right)\mathrm{d}\alpha.$$
- By orthogonality, the quantity T is bounded above by the number of integral solutions of the equation
$$\sum_{i=1}^{2^{J-1}} \left(u_i^k - v_i^k\right) = \Delta_J\left(x^k;\mathbf{h}\right),$$
with $1 \leqslant u_i, v_i \leqslant X$ for $i \in [2^{J-1}]$, $1 \leqslant x \leqslant X$ and $|h_j| < X$ for $j \in [J]$.

Proof

- Here that we have weakened the condition $x \in I_J(\mathbf{h})$ to $1 \leqslant x \leqslant X$. This may increase the potential number of solutions.
- The solutions counted by T are of two types. First, there are the solutions in which
$$\sum_{i=1}^{2^{J-1}} \left(u_i^k - v_i^k\right) = 0.$$
- In this situation, one has $\Delta_J\left(x^k; \mathbf{h}\right) = 0$. By orthogonality, the number of choices of $\mathbf{u}$ and $\mathbf{v}$ here is
$$\int_0^1 f(\alpha)^{2^{J-1}} f(-\alpha)^{2^{J-1}} \, \mathrm{d}\alpha = \int_0^1 |f(\alpha)|^{2^J} \, \mathrm{d}\alpha = O\left(X^{2^J - J + \varepsilon}\right),$$
by invoking the inductive hypothesis.
- Since $\Delta_J\left(x^k; \mathbf{h}\right) = 0$, we have
$$h_1 \ldots h_J \left(\frac{k!}{(k-J)!} x^{k-J} + \ldots\right) = 0.$$
- So either $h_j = 0$ for some $j \in [J]$, or else x satisfies a polynomial equation determined by $h_1, \ldots, h_J$.

Proof

- Then the total number of choices for x and $h_1, \ldots, h_J$ is $O\left(X^J\right)$. The contribution of the solutions of this first type to T is consequently

$$O\big(X^J \cdot X^{2^J - J + \varepsilon}\big) = O\big(X^{2^J + \varepsilon}\big).$$

- For the second class of solutions counted by T, one has

$$\sum_{i=1}^{2^{J-1}} \left(u_i^k - v_i^k\right) = N$$

  for some non-zero integer $N = N(\mathbf{u}, \mathbf{v})$ with $|N| = O(X^k)$.
- For each such choice of $\mathbf{u}$ and $\mathbf{v}$, we have

$$h_1 \ldots h_J \left(\frac{k!}{(k-J)!} x^{k-J} + \ldots\right) = N$$

  and thus there are $O\left(\tau_{J+1}(N)\right) = O\big(X^{\varepsilon}\big)$ possible choices for $h_1, \ldots, h_J$ and x. The contribution to T from this second class of solutions is therefore

$$O\Big(\sum_{\substack{1 \leqslant u_i, v_i \leqslant X \\ 1 \leqslant i \leqslant 2^{J-1}}} \tau_{J+1}(N(\mathbf{u}, \mathbf{v}))\Big) = O\Big(X^{\varepsilon} \sum_{\substack{1 \leqslant u_i, v_i \leqslant X \\ 1 \leqslant i \leqslant 2^{J-1}}} 1\Big) = O\big(X^{2^J + \varepsilon}\big).$$

Proof

- Combining these two estimates, we see that $T = O\big(X^{2^J+\varepsilon}\big)$.
- Substituting this bound into (*), which we recall

$$\int_0^1 |f(\alpha)|^{2^{J+1}}\, d\alpha = \int_0^1 f(\alpha)^{2^{J-1}} f(-\alpha)^{2^{J-1}} |f(\alpha)|^{2^J}\, d\alpha \leqslant (2X)^{2^J-J-1} T, \qquad (*)$$

  we conclude that

$$\int_0^1 |f(\alpha)|^{2^{J+1}}\, d\alpha = O\big((2X)^{2^J-J-1} \cdot X^{2^J+\varepsilon}\big) = O\big(X^{2^{J+1}-(J+1)+\varepsilon}\big).$$

- This proves the case $j = J+1$, and the conclusion of the lemma follows. □