Analytic Number Theory
Lecture 13

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Ternary Goldbach problem

The counting function for the number of representations of an odd integer N as the sum of three primes is

$$r(N) = \sum_{p_1+p_2+p_3=N} 1.$$

The following is Vinogradov's asymptotic formula for $r(N)$.

Theorem (Vinogradov)

There exists an arithmetic function $\mathfrak{S}(N)$ and $c_1, c_2 \in \mathbb{R}_+$ such that

$$c_1 < \mathfrak{S}(N) < c_2$$

for all sufficiently large odd integers N, and

$$r(N) = \mathfrak{G}(N) \frac{N^2}{2(\log N)^3} \left(1 + O\left(\frac{\log\log N}{\log N}\right)\right).$$

The arithmetic function $\mathfrak{S}(N)$ is called the singular series for the ternary Goldbach problem.

The singular series

- We begin by studying the arithmetic function

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{\mu(q)c_q(N)}{\varphi(q)^3},$$

  where

$$c_q(N) = \sum_{\substack{a=1 \\ (q,a)=1}}^{q} e(aN/q)$$

  is Ramanujan's sum.
- The function $\mathfrak{S}(N)$ is called the singular series for the ternary Goldbach problem.
- The Ramanujan sum $c_q(n)$ is a multiplicative function of q; that is, if $(q, q') = 1$, then

$$c_{qq'}(n) = c_q(n) \cdot c_{q'}(n).$$

- The Ramanujan sum can be expressed in the form

$$c_q(n) = \sum_{d|(q,n)} \mu\left(\frac{q}{d}\right) d,$$

- In particular, if $(q, n) = 1$, then $c_q(n) = \mu(q)$.

The singular series

Theorem

The singular series $\mathfrak{S}(N)$ converges absolutely and uniformly in N and has the Euler product representation

$$\mathfrak{S}(N) = \prod_{p} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|N} \left(1 - \frac{1}{p^2 - 3p + 3}\right).$$

There exist positive constants c_1 and c_2 such that

$$c_1 < \mathfrak{S}(N) < c_2,$$

for all positive integers N. Moreover, for any $\varepsilon > 0$,

$$\mathfrak{S}(N,Q) = \sum_{q \leq Q} \frac{\mu(q) c_q(N)}{\varphi(q)^3} = \mathfrak{S}(N) + O\left(Q^{-(1-\varepsilon)}\right),$$

where the implied constant depends only on $\varepsilon \in (0,1)$.

Decomposition into major and minor arcs

- We decompose the unit interval $[0,1]$ into two disjoint sets: the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$.
- Let $B > 0$ and set
$$Q = (\log N)^B$$
- For $1 \leq q \leq Q$ and $0 \leq a \leq q$ satisfying $(a,q) = 1$, the major arc $\mathfrak{M}(q,a)$ is the interval consisting of all real numbers $\alpha \in [0,1]$ so that
$$\left|\alpha - \frac{a}{q}\right| \leq \frac{Q}{N}.$$
- If $\alpha \in \mathfrak{M}(q,a) \cap \mathfrak{M}(q',a')$ and $a/q \neq a'/q'$, then $|aq' - a'q| \geq 1$ and
$$\begin{aligned} \frac{1}{Q^2} \leq \frac{1}{qq'} \leq \frac{|aq' - a'q|}{qq'} &= \left|\frac{a}{q} - \frac{a'}{q'}\right| \\ &\leq \left|\frac{a}{q} - \alpha\right| + \left|\alpha - \frac{a'}{q'}\right| \leq \frac{2Q}{N} \end{aligned}$$
or, equivalently,
$$N \leq 2Q^3 = 2(\log N)^{3B},$$
- This is impossible for N sufficiently large.

Decomposition into major and minor arcs

- Therefore, the major arcs $\mathfrak{M}(q,a)$ are pairwise disjoint for large N. The set of major arcs is

$$\mathfrak{M} = \bigcup_{q=1}^{Q} \bigcup_{\substack{a=0 \\ (a,q)=1}}^{q} \mathfrak{M}(q,a) \subseteq [0,1],$$

  and the set of minor arcs is

$$\mathfrak{m} = [0,1] \backslash \mathfrak{M},$$

- We consider a weighted sum over the representations of N as a sum of three primes:

$$R(N) = \sum_{p_1+p_2+p_1=N} \log p_1 \log p_2 \log p_3$$

- Vinogradov obtained an asymptotic formula for $R(N)$, from which the ternary Goldbach problem will follow by an elementary argument.

Circle method

- We can use the circle method to express the representation function $R(N)$ as the integral of a trigonometric polynomial over the major and minor arcs. Let
$$F(\alpha) = \sum_{p \le N} (\log p) e(p\alpha).$$
- This exponential sum over primes is the generating function for $R(N)$:
$$\begin{aligned} R(N) &= \sum_{p_1+p_2+p_3=N} \log p_1 \log p_2 \log p_3 \\ &= \int_0^1 F(\alpha)^3 e(-N\alpha) d\alpha \\ &= \int_{\mathfrak{M}} F(\alpha)^3 e(-N\alpha) d\alpha + \int_{\mathfrak{m}} F(\alpha)^3 e(-N\alpha) d\alpha. \end{aligned}$$
- The main term in Vinogradov's theorem will come from the integral over the major arcs, and the integral over the minor arcs will be negligible.
- Just as in the Hardy–Littlewood asymptotic formula, the integral over the major arcs in Vinogradov's theorem is (except for a small error term) the product of the singular series $\mathfrak{S}(N)$ and an integral $J(N)$. In this case, the integral $J(N)$ is very easy to evaluate.

The integral over the major arcs

Lemma

Let

$$u(\beta) = \sum_{m=1}^{N} e(m\beta).$$

Then

$$J(N) = \int_{-1/2}^{1/2} u(\beta)^3 e(-N\beta) d\beta = \frac{N^2}{2} + O(N).$$

Proof.

The number of representations of N as the sum of three positive integers is

$$\begin{aligned} J(N) &= \int_{-1/2}^{1/2} u(\beta)^3 e(-N\beta) d\beta \\ &= \int_{-1/2}^{1/2} \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} \sum_{m_3=1}^{N} e\left((m_1 + m_2 + m_3 - N)\, \beta \right) d\beta \\ &= \binom{N-1}{2} = \frac{N^2}{2} + O(N). \qquad \square \end{aligned}$$

Siegel–Walfisz and approximations

Theorem

If $q \in \mathbb{Z}_+$ and $(q,a) = 1$, then, for any $C > 0$, one has

$$\vartheta(x;q,a) = \sum_{\substack{p \le x \\ p = a \pmod q}} \log p = \frac{x}{\varphi(q)} + O\left(\frac{x}{(\log x)^C}\right)$$

for all $x \ge 2$, where the implied constant depends only on C.

Lemma

Let

$$F_x(\alpha) = \sum_{p=1} (\log p) e(p\alpha).$$

Let $B, C \in \mathbb{R}_+$. If $1 \le q \le Q = (\log N)^B$ and $(q,a) = 1$, then

$$F_x(a/q) = \frac{\mu(q)}{\varphi(q)} x + O\left(\frac{QN}{(\log N)^C}\right)$$

for $1 \le x \le N$, where the implied constant depends only on B and C.

Further approximations

Remember that

$$u(\beta) = \sum_{m=1}^{N} e(m\beta).$$

Lemma

Let $B, C \in \mathbb{R}_+$ be such that $C > 2B$. If $\alpha \in \mathfrak{M}(q,a)$ and $\beta = \alpha - a/q$, then

$$F(\alpha) = \frac{\mu(q)}{\varphi(q)} u(\beta) + O\left(\frac{Q^2 N}{(\log N)^C}\right),$$

and

$$F(\alpha)^3 = \frac{\mu(q)}{\varphi(q)^3} u(\beta)^3 + O\left(\frac{Q^2 N^3}{(\log N)^C}\right),$$

where the implied constants depend only on B and C.

Contribution on major arcs

Theorem

For any $B, C, \varepsilon \in \mathbb{R}_+$ *with* $C > 2B$*, the integral over the major arcs is*

$$\int_{\mathfrak{M}} F(\alpha)^3 e(-N\alpha)\,d\alpha = \mathfrak{S}(N)\frac{N^2}{2} + O\left(\frac{N^2}{(\log N)^{(1-\varepsilon)B}}\right) + O\left(\frac{N^2}{(\log N)^{C-5B}}\right),$$

where the implied constants depend only on B, C*, and* ε*.*

An exponential sum over primes

Recall that

$$F(\alpha) = \sum_{p \leq N} (\log p) e(p\alpha).$$

Theorem (Vinogradov's inequality)

If

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2},$$

where $a, q \in \mathbb{Z}$ *are such that* $1 \leq q \leq N$ *and* $(a, q) = 1$, *then*

$$|F(\alpha)| = O\left(\left(\frac{N}{q^{1/2}} + N^{4/5} + N^{1/2} q^{1/2} \right) (\log N)^4 \right).$$

Vaughan's identity

Lemma (Vaughan's identity)

For $u \geq 1$, let

$$M_u(k) = \sum_{\substack{d,k \\ d \leq u}} \mu(d).$$

Let $\Phi(k,l)$ be an arithmetic function of two variables. Then

$$\sum_{u<l\leq N} \Phi(1,l) + \sum_{u<k\leq N} \sum_{u<l\leq \frac{N}{k}} M_u(k)\Phi(k,l) = \sum_{d\leq u} \sum_{u<l\leq \frac{N}{d}} \sum_{m\leq \frac{N}{ld}} \mu(d)\Phi(dm,l).$$

Proof.

Hint: evaluate the sum

$$S = \sum_{k=1}^{N} \sum_{u<l\leq \frac{N}{k}} M_u(k)\Phi(k,l)$$

in two different ways. □

Vaughan's identity in estimates of exponential sums

Lemma

Let $\Lambda(l)$ be the von Mangoldt function. For every $\alpha \in \mathbb{R}$, one has

$$F(\alpha) = \sum_{p \leq N} (\log p) e(p\alpha) = S_1 - S_2 - S_3 + O\big(N^{1/2}\big),$$

where

$$S_1 = \sum_{d \leq N^{2/5}} \sum_{l \leq \frac{N}{d}} \sum_{m \leq \frac{N}{ld}} \mu(d)\Lambda(l)e(\alpha dlm),$$

$$S_2 = \sum_{d \leq N^{2/5}} \sum_{l \leq N^{2/5}} \sum_{m \leq \frac{N}{ld}} \mu(d)\Lambda(l)e(\alpha dlm),$$

and

$$S_3 = \sum_{k > N^{2/S}} \sum_{N^{2/5} < l \leq N/k} M_{N^{2/5}}(k)\Lambda(l)e(\alpha kl).$$

Proof.

Apply Vaughan's identity with $u = N^{2/5}$ and $\Phi(k,l) = \Lambda(l)e(\alpha kl)$, and note

$$\sum_{u < l \leq N} \Phi(1,l) = F(\alpha) + O\big(N^{1/2}\big). \qquad \square$$

Proof of Vinogradov's inequality

Lemma

If

$$\left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^2},$$

where $1 \le q \le N$ *and* $(a,q) = 1$, *then*

$$|S_1| = O\Big(\left(\frac{N}{q} + N^{2/5} + q \right) (\log N)^2 \Big).$$

Lemma

If

$$\left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^2},$$

where $1 \le q \le N$ *and* $(a,q) = 1$, *then*

$$|S_2| = O\Big(\left(\frac{N}{q} + N^{4/5} + q \right) (\log N)^2 \Big).$$

Proof of Vinogradov's inequality

Lemma

If

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^2},$$

where $1 \leq q \leq N$ *and* $(a,q) = 1$*, then*

$$|S_3| = O\left(\left(\frac{N}{q^{1/2}} + N^{4/5} + N^{1/2}q^{1/2}\right)(\log N)^4\right).$$

Theorem

For any $B > 0$*, we have*

$$\left|\int_{\mathfrak{m}} F(\alpha)^3 e(-\alpha N) d\alpha\right| = O\left(\frac{N^2}{(\log N)^{(B/2)-5}}\right).$$

where the implied constant depends only on B.

Proof

- Recall that $Q = (\log N)^B$. Let $\alpha \in \mathfrak{m} = [0,1] \backslash \mathfrak{M}$. By Dirichlet's theoremfor any real number α there exists a fraction $a/q \in [0,1]$ with $1 \le q \le N/Q$ and $(a,q) = 1$ such that

$$\left| \alpha - \frac{a}{q} \right| \le \frac{Q}{qN} \le \min\left(\frac{Q}{N}, \frac{1}{q^2} \right).$$

- If $q \le Q$, then $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, which is false. Therefore,

$$Q < q \le \frac{N}{Q}.$$

- Then by the previous theorem we obtain

$$\begin{aligned} |F(\alpha)| &= O\Bigg(\left(\frac{N}{q^{1/2}} + N^{4/5} + N^{1/2} q^{1/2} \right) (\log N)^4 \Bigg) \\ &= O\Bigg(\left(\frac{N}{(\log N)^{B/2}} + N^{4/5} + N^{1/2} \left(\frac{N}{(\log N)^B} \right)^{1/2} \right) (\log N)^4 \Bigg) \\ &= O\Bigg(\frac{N}{(\log N)^{(B/2)-4}} \Bigg). \end{aligned}$$

Proof

- Since
$$\vartheta(N) = \sum_{p \le N} \log p = O(N),$$
we have
$$\int_0^1 |F(\alpha)|^2 d\alpha = \sum_{p \le N} (\log p)^2 \le \log N \sum_{p \le N} \log p = O(N \log N).$$
- Thus
$$\begin{aligned}\int_{\mathfrak{m}} |F(\alpha)|^3 d\alpha &= O\left(\sup\{|F(\alpha)| : \alpha \in \mathfrak{m}\} \int_{\mathfrak{m}} |F(\alpha)|^2 d\alpha \right) \\ &= O\left(\frac{N}{(\log N)^{(B/2)-4}} \int_0^1 |F(\alpha)|^2 d\alpha \right) \\ &= O\left(\frac{N^2}{(\log N)^{(B/2)-5}} \right).\end{aligned}$$
This completes the proof. □

Vinogradov's theorem

Theorem (Vinogradov)

Let $\mathfrak{S}(N)$ be the singular series for the ternary Goldbach problem, i.e.

$$\mathfrak{S}(N) = \prod_{p}\left(1 + \frac{1}{(p-1)^3}\right)\prod_{p|N}\left(1 - \frac{1}{p^2 - 3p + 3}\right).$$

- *For all sufficiently large odd integers N and for every $A > 0$, we have*

$$R(N) = \sum_{p_1+p_2+p_1=N} \log p_1 \log p_2 \log p_3 = \mathfrak{S}(N)\frac{N^2}{2} + O\left(\frac{N^2}{(\log N)^A}\right),$$

  where the implied constant depends only on A.

- *In particular, we have*

$$r(N) = \sum_{p_1+p_2+p_3=N} 1 = \mathfrak{S}(N)\frac{N^2}{2(\log N)^3}\left(1 + O\left(\frac{\log\log N}{\log N}\right)\right).$$