Analytic Number Theory
Lecture 3

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Monotonic, unimodal and integrable functions

Definition

A function $f : I \to \mathbb{R}$ is unimodal on an interval $I = [a, b]$ if there exists a number $t_0 \in I$ such that $f(t)$ is increasing for $t \leq t_0$ and decreasing for $t \geq t_0$. For example, $f(t) = \log^k t/t$ is unimodal on the interval $[1, \infty)$ with $t_0 = e^k$.

Theorem

- *Let $a, b \in \mathbb{Z}$with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be monotonic. Then*

$$\min\{f(a), f(b)\} \leq \sum_{n=a}^{b} f(n) - \int_a^b f(t)dt \leq \max\{f(a), f(b)\}.$$

- *Let $x, y \in \mathbb{R}$ with $y < \lfloor x \rfloor$, and let $f : [y, x] \to \mathbb{R}_+$ be monotonic. Then*

$$\left| \sum_{y<n\leq x} f(n) - \int_y^x f(t)dt \right| \leq \max\{f(y), f(x)\}.$$

- *Let $f : [1, \infty) \to \mathbb{R}_+$ be unimodal. Then*

$$F(x) = \sum_{n\leq x} f(n) = \int_1^x f(t)dt + O(1).$$

Proof

- If $f : [a, b] \to \mathbb{R}$ is increasing, then

$$\int_a^b f(t)dt = \sum_{k=a}^{b-1} \int_k^{k+1} f(t)dt \leq \sum_{k=a+1}^{b} f(k),$$

and also

$$\int_a^b f(t)dt = \sum_{k=a}^{b-1} \int_k^{k+1} f(t)dt \geq \sum_{k=a}^{b-1} f(k),$$

- Hence, we conclude that

$$f(a) + \int_a^b f(t)dt \leq \sum_{k=a}^{b} f(k) \leq f(b) + \int_a^b f(t)dt.$$

- If $f : [a, b] \to \mathbb{R}$ is decreasing, then

$$f(b) + \int_a^b f(t)dt \leq \sum_{k=a}^{b} f(k) \leq f(a) + \int_a^b f(t)dt.$$

- Combining those two inequalities we obtain the desired conclusion.

Proof

- Let $f : [y,x] \to \mathbb{R}_+$ be increasing. Let $a = \lfloor y \rfloor + 1$ and $b = \lfloor x \rfloor$. We have $y < a \le b \le x$ and

$$\sum_{y<n\le x} f(n) = \sum_{a\le n\le b} f(n) \le \int_a^b f(t)dt + f(b) \le \int_y^x f(t)dt + f(x)$$

- Since $f(a) \ge \int_y^a f(t)dt$ and $f(x) \ge \int_b^x f(t)dt$, it follows that

$$\begin{aligned}\sum_{y<n\le x} f(n) &\ge \int_a^b f(t)dt + f(a) \\ &\ge \int_y^x f(t)dt - \int_b^x f(t)dt + f(a) - \int_y^a f(t)dt \\ &\ge \int_y^x f(t)dt - f(x)\end{aligned}$$

- Therefore,

$$\left| \sum_{y<n\le x} f(n) - \int_x^y f(t)dt \right| \le f(x).$$

Proof

- If $f : [y, x] \to \mathbb{R}_+$ be decreasing, we obtain a similar conclusion
$$\left| \sum_{y<n\leq x} f(n) - \int_x^y f(t)dt \right| \leq f(y),$$
which proves the second inequality.
- Finally, if the function $f : [1, \infty) \to \mathbb{R}_+$ is unimodal, then $f(t)$ is bounded and the conclusion follows from the second inequality. □

Example: a simple form of Stirling's formula

For $x \geq 2$, we have

$$\sum_{n\leq x} \log n = x \log x - x + 1 + O(\log x).$$

- Indeed, the function $f(t) = \log t$ is increasing on $[1, x]$. By the previous theorem, we have
$$\int_1^x \log t dt \leq \sum_{n\leq x} \log n \leq \int_1^x \log t dt + \log x,$$
which gives the desired claim, since $\int_1^x \log t dt = x \log x - x + 1$.

Stirling's formula

Theorem

For $n \in \mathbb{N}$, we have

$$n! = \sqrt{2\pi} n^{n+1/2} e^{-n} e^{r_n},$$

where r_n satisfies the double inequality

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}.$$

In other words we have

$$\sqrt{2\pi} n^{n+1/2} e^{-n} e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi} n^{n+1/2} e^{-n} e^{\frac{1}{12n}}.$$

Proof

- Let

$$S_n = \log(n!) = \sum_{p=1}^{n-1} \log(p+1)$$

and write

$$\log(p+1) = A_p + b_p - \varepsilon_p,$$

where

$$\begin{aligned} A_p &= \int_p^{p+1} \log x \, dx, \\ b_p &= [\log(p+1) - \log p]/2, \\ \varepsilon_p &= \int_p^{p+1} \log x \, dx - [\log(p+1) + \log p]/2. \end{aligned}$$

- In other words, $\log(p+1)$ is regarded as the area of a rectangle with base $(p, p+1)$ and height $\log(p+1)$ partitioned into a curvilinear area A_p, a triangle b_p, and a small sliver ε_p suggested by the geometry of the curve $y = \log x$.

Proof

- Then

$$S_n = \sum_{p=1}^{n-1}(A_p + b_p - \varepsilon_p) = \int_1^n \log x\,dx + \frac{1}{2}\log n - \sum_{p=1}^{n-1}\varepsilon_p.$$

- Since $\int \log x\,dx = x\log x - x$ we can write

$$S_n = (n + 1/2)\log n - n + 1 - \sum_{p=1}^{n-1}\varepsilon_p,$$

where

$$\begin{aligned}\varepsilon_p &= \int_p^{p+1} \log x\,dx - [\log(p+1) + \log p]/2 \\ &= (p+1)\log(p+1) - p\log p - 1 - [\log(p+1) + \log p]/2 \\ &= \frac{2p+1}{2}\log\left(\frac{p+1}{p}\right) - 1.\end{aligned}$$

Proof

- Using the well known series expansions

$$\log\left(\frac{1+x}{1-x}\right) = 2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$$

valid for $|x| < 1$, and setting $x = (2p+1)^{-1}$, so that $(1+x)/(1-x) = (p+1)/p$, we find that

$$\varepsilon_p = \frac{2p+1}{2}\log\left(\frac{p+1}{p}\right) - 1 = \sum_{k=0}^{\infty}\frac{1}{(2k+3)(2p+1)^{2k+2}}.$$

- We can therefore bound ε_p above:

$$\varepsilon_p < \frac{1}{3(2p+1)^2}\sum_{k=0}^{\infty}\frac{1}{(2p+1)^{2k}} = \frac{1}{12}\left(\frac{1}{p} - \frac{1}{p+1}\right).$$

Proof

- Similarly, we bound ε_p below:

$$\varepsilon_p > \frac{1}{3(2p+1)^2} \sum_{k=0}^{\infty} \frac{1}{[3(2p+1)^2]^k} = \frac{1}{3(2p+1)^2} \frac{1}{1 - \frac{1}{3(2p+1)^2}}$$
$$> \frac{1}{12}\left(\frac{1}{p+1/12} - \frac{1}{p+1+1/12}\right).$$

- Now define

$$B = \sum_{p=1}^{\infty} \varepsilon_p, \qquad r_n = \sum_{p=n}^{\infty} \varepsilon_p,$$

where from the lower and upper bound for ε_p we have

$$1/13 < B < 1/12.$$

- Then we can write

$$S_n = (n + 1/2)\log n - n + 1 - \sum_{p=1}^{n-1} \varepsilon_p$$
$$= (n + 1/2)\log n - n + 1 - B + r_n,$$

or, setting $C = e^{1-B}$, as

$$n! = Cn^{n+1/2}e^{-n}e^{r_n},$$

where r_n satisfies

$$1/(12n+1) < r_n < 1/(12n).$$

- The constant C, lies between $e^{11/12}$ and $e^{12/13}$, may be shown to have the value $\sqrt{2\pi}$. Indeed, by the Wallis formula we have

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty} \frac{(2^n n!)^2}{(2n)!\sqrt{2n}} = \lim_{n\to\infty} \frac{C^2 2^{2n} n^{2n+1} e^{-2n}}{C(2n)^{2n+1/2}e^{-2n}\sqrt{2n}} = \frac{C}{2}.$$

- Thus $C = \sqrt{2\pi}$ and this completes the proof. □

Partial summation

Theorem

- *Let $f, g : \mathbb{Z}_+ \to \mathbb{C}$ be arithmetic functions. Let $F(x) := \sum_{1 \le n \le x} f(n)$. Then for any $a, b \in \mathbb{N}$ with $a < b$, we have*

$$\sum_{n=a+1}^{b} f(n)g(n) = F(b)g(b) - F(a)g(a+1) - \sum_{n=a+1}^{b-1} F(n)(g(n+1) - g(n)).$$

- *Let $x, y \in \mathbb{R}_+$ with $\lfloor y \rfloor < \lfloor x \rfloor$, and let $g \in C^1([y, x])$. Then*

$$\sum_{y<n\le x} f(n)g(n) = F(x)g(x) - F(y)g(y) - \int_y^x F(t)g'(t)dt$$

- *In particular, if $x \ge 2$ and $g \in C^1([1, x])$, then*

$$\sum_{n\le x} f(n)g(n) = F(x)g(x) - \int_1^x F(t)g'(t)dt$$

Proof

- Since $f(n) = F(n) - F(n-1)$, we have

$$\begin{aligned}\sum_{n=a+1}^{b} f(n)g(n) &= \sum_{n=a+1}^{b} (F(n) - F(n-1))g(n) \\ &= \sum_{n=a+1}^{b} F(n)g(n) - \sum_{n=a}^{b-1} F(n)g(n+1) \\ &= F(b)g(b) - F(a)g(a+1) - \sum_{n=a+1}^{b-1} F(n)(g(n+1) - g(n)).\end{aligned}$$

- If $g \in C^1([y,x])$, then

$$g(n+1) - g(n) = \int_n^{n+1} g'(t)dt,$$

and since $F(t) = F(n)$ for $n \leq t < n+1$, it follows that

$$F(n)(g(n+1) - g(n)) = \int_n^{n+1} F(t)g'(t)dt.$$

Proof

- Let $a = \lfloor y \rfloor, b = \lfloor x \rfloor$, since $a \leq y < a+1 \leq b \leq x < b+1$, we have

$$\sum_{y<n\leq x} f(n)g(n) = \sum_{n=a+1}^{b} f(n)g(n)$$

$$= F(b)g(b) - F(a)g(a+1) - \sum_{n=a+1}^{b-1} F(n)(g(n+1) - g(n))$$

$$= F(x)g(b) - F(y)g(a+1) - \sum_{n=a+1}^{b-1} \int_{n}^{n+1} F(t)g'(t)dt$$

$$= F(x)g(x) - F(y)g(y) - F(x)(g(x) - g(b)) - F(y)(g(a+1) - g(y))$$

$$- \int_{a+1}^{b} F(t)g'(t)dt = F(x)g(x) - F(y)g(y) - \int_{y}^{x} F(t)g'(t)dt.$$

- If $x \geq 2$ and $g \in C^1([1,x])$, then

$$\sum_{n\leq x} f(n)g(n) = f(1)g(1) + \sum_{1<n\leq x} f(n)g(n)$$

$$= f(1)g(1) + F(x)g(x) - F(1)g(1) - \int_{1}^{x} F(t)g'(t)dt$$

$$= F(x)g(x) - \int_{1}^{x} F(t)g'(t)dt. \qquad \square$$

Partial summation $\equiv$ Abel summation formula

Let $(a_k)_{k\in\mathbb{Z}}, (b_k)_{k\in\mathbb{Z}} \subseteq \mathbb{C}$ and $m, n \in \mathbb{Z}$ with $m < n$. If $s_k := \sum_{l=m}^{k} a_l$, then

$$\sum_{k=m+1}^{n} a_k b_k = s_n b_n - a_m b_{m+1} - \sum_{k=m+1}^{n-1} (b_{k+1} - b_k)\, s_k.$$

- Observe that $a_k = s_k - s_{k-1}$, hence

$$\sum_{k=m+1}^{n} a_k b_k = \sum_{k=m+1}^{n} b_k\,(s_k - s_{k-1}) = \sum_{k=m+1}^{n} b_k s_k - \sum_{k=m}^{n-1} b_{k+1} s_k,$$

  which implies the asserted result, since $s_m = a_m$.
- One can deduce from this equality the following useful bound

$$\Big|\sum_{k=m+1}^{n} a_k b_k\Big| \leq (2\max\{|b_{m+1}|, |b_n|\} + V_{[m,n]}) \max_{m\leqslant k\leqslant n} |s_k|,$$

  where $V_{[m,n]} := \sum_{k=m+1}^{n-1} |b_{k+1} - b_k|$.
- If $(b_k)_{k\in\mathbb{Z}} \subseteq \mathbb{R}_+$ is a monotone, then

$$\Big|\sum_{k=m+1}^{n} a_k b_k\Big| \leq 2\max\{b_{m+1}, b_n\} \max_{m\leqslant k\leqslant n} |s_k|.$$

Euler–Mascheroni constant constant

Since

$$\sum_{k=1}^{n} \frac{1}{k} \geqslant \int_1^n \frac{\mathrm{d}t}{t} = \log n,$$

the sequence

$$u_n := \sum_{k=1}^{n} \frac{1}{k} - \log n \geq 0$$

is non-negative, and decreasing

$$u_{n+1} - u_n = \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right) \leqslant 0$$

thus $(u_n)_{n\in\mathbb{Z}_+}$ converges.

Definition

We call the Euler-Mascheroni constant, or more simply the Euler constant, the real number γ defined by

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) \approx 0.577215664901532\ldots.$$

The problem of the irrationality of γ still remains open!

Integral representation of γ

Theorem

Let $n \in \mathbb{Z}_+$, then

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + R(n)$$

with $0 \leqslant R(n) < \frac{1}{n}$.

Proof.

- We use partial summation with $f(t) = 1$ and $g(t) = \frac{1}{t}$ which gives

$$\sum_{k=1}^{n} \frac{1}{k} = \frac{1}{n} \sum_{k=1}^{n} 1 + \int_1^n \frac{1}{t^2} \Big(\sum_{1 \leq k \leqslant t} 1 \Big) dt$$
$$= 1 + \int_1^n \frac{t - \{t\}}{t^2} dt = \log n + \left(1 - \int_1^\infty \frac{\{t\}}{t^2} dt \right) + \int_n^\infty \frac{\{t\}}{t^2} dt$$

- Thus $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 1 - \int_1^\infty \frac{\{t\}}{t^2} dt$, and

$$R(n) := \int_n^\infty \frac{\{t\}}{t^2} dt < \int_n^\infty \frac{1}{t^2} dt = \frac{1}{n}. \qquad \square$$

Absolute convergence of Dirichlet series

Proposition

For each Dirichlet series $D(s,f)$, there exists $\sigma_a \in \mathbb{R} \cup \{\pm\infty\}$, called the abscissa of absolute convergence, such that

- *$D(s,f)$ converges absolutely in the half-plane $\sigma > \sigma_a$;*
- *$D(s,f)$ does not converge absolutely in the half-plane $\sigma < \sigma_a$.*

Remarks

- In particular, the series $D(s,f)$ defines an analytic function in the halfplane $\sigma > \sigma_a$. By abuse of notation, this function will be still denoted by $D(s,f)$.
- If $|f(n)| \leqslant \log n$, then the series $D(s,f)$ is absolutely convergent in the half-plane $\sigma > 1$, and hence $\sigma_a \leqslant 1$.
- At $\sigma = \sigma_a$, the series may or may not converge absolutely. For instance, $\zeta(s)$ converges absolutely in the half-plane $\sigma > \sigma_a = 1$, but does not converge on the line $\sigma = 1$.
- On the other hand, the Dirichlet series associated to the function $f(n) = 1/(\log(en))^2$ has also $\sigma_a = 1$ for the abscissa of absolute convergence, but converges absolutely at $\sigma = 1$.

Proof

- Let $S := \{s \in \mathbb{C} : D(s,f) \text{ converges absolutely}\}$.
- If $S = \varnothing$, then put $\sigma_a = +\infty$. Otherwise define

$$\sigma_a := \inf\{\sigma : s = \sigma + it \in S\}$$

- $D(s,f)$ does not converge absolutely if $\sigma < \sigma_a$ by the definition of σ_a.
- On the other hand, suppose that $D(s,f)$ is absolutely convergent for some $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ and let $s = \sigma + it$ be such that $\sigma \geqslant \sigma_0$. Since

$$\left|\frac{f(n)}{n^s}\right| = \left|\frac{f(n)}{n^{s_0}}\right| \times \frac{1}{n^{\sigma-\sigma_0}} \leqslant \left|\frac{f(n)}{n^{s_0}}\right|$$

  we infer that $D(s,f)$ converges absolutely at any point s with $\sigma \geqslant \sigma_0$.
- Now by the definition of σ_a, there exist points arbitrarily close to σ_a at which $D(s,f)$ converges absolutely, and therefore by above $D(s,f)$ converges absolutely at each point s such that $\sigma > \sigma_a$. □

The partial sums $\sum_{x<n\leqslant y} f(n)$ and the Dirichlet series $D(s,f)$ are strongly related to each other. The next result shows that if we are able to estimate the order of magnitude of $\sum_{x<n\leqslant y} f(n)$, then a region of absolute convergence of $D(s,f)$ is known.

Simple criterium for absolute convergence

Proposition

Let $D(s,f) = \sum_{n=1}^{\infty} f(n)n^{-s}$ *be a Dirichlet series. Assume that*

$$|f(n)| \leqslant Mn^{\alpha} \quad \textit{for all} \quad n \in \mathbb{Z}_+,$$

for some $\alpha \geqslant 0$ *and* $M > 0$ *independent of* n.

- *Then* $D(s,f)$ *converges absolutely in the half-plane* $\sigma > \alpha + 1$.

Proof.

Indeed, observe that

$$|D(s,f)| = \sum_{n=1}^{\infty} |f(n)n^{-s}| \leq M \sum_{n=1}^{\infty} n^{-(\sigma-\alpha)} < \infty,$$

whenever $\sigma > \alpha + 1$, as desired. □

Dirichlet series for products

Proposition

Let f, g be two arithmetic functions. The the associated Dirichlet series $D(s,f)$ and $D(s,g)$ are absolutely convergent at a point s_0, then $D(s,f\star g)$ converges absolutely at s_0 and we have $D(s_0,f\star g) = D(s_0,f)D(s_0,g)$.

Proof.

- We have

$$D(s_0,f)D(s_0,g) = \sum_{n=1}^{\infty} \frac{f\star g(n)}{n^{s_0}} = D(s,f\star g),$$

  where the rearrangement of the terms in the double sums is justified by the absolute convergence of the two series $D(s,f)$ and $D(s,g)$ at $s = s_0$.

- Furthermore, we have

$$\sum_{n=1}^{\infty} \left|\frac{f\star g(n)}{n^{s_0}}\right| \leqslant \left(\sum_{n=1}^{\infty} \left|\frac{f(n)}{n^{s_0}}\right|\right) \left(\sum_{n=1}^{\infty} \left|\frac{g(n)}{n^{s_0}}\right|\right)$$

  proving the absolute convergence of $D(s_0,f\star g)$.

This completes the proof. □

Dirichlet series for inverses

Corollary

Let f be an arithmetic function such that $f(1) \neq 0$. Let f^{-1} be the convolution inverse of the fuction f, i.e. $f \star f^{-1} = \delta$. Then

$$D(s, f^{-1}) = \frac{1}{D(s, f)}$$

at every point s where $D(s, f)$ and $D(s, f^{-1})$ converge absolutely.

Example

- *The Möbius function satisfies $\mu^{-1} = \mathbf{1}$, hence for $\sigma > 1$, we have*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

- *In particular, we have*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}, \quad \textit{since} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Computation of abscissa of the absolute convergence

Proposition

Let f be an arithmetic function.

(i) *If $|f(n)| \leqslant Mn^{\alpha}$ for some $M \in \mathbb{R}_+$ and $\alpha \geqslant 0$, then $\sigma_a \leqslant \alpha + 1$.*

(ii) *We have*

$$\sigma_a \leq L := \limsup_{n \to \infty} \left(1 + \frac{\log |f(n)|}{\log n}\right).$$

Proof.

- The first part follows from our criterium for absolute convergence.
- For the second part we may suppose that $L < \infty$ and let $\sigma > L$. Fix $\varepsilon > 0$ so that $\sigma - \varepsilon > L + \varepsilon$. Then there exists a large $N_\varepsilon \in \mathbb{Z}_+$ such that, for all $n \geqslant N_\varepsilon$, we have

$$1 + \frac{\log |f(n)|}{\log n} < L + \varepsilon < \sigma - \varepsilon$$

  and hence, for all $n \geqslant N_\varepsilon$, we obtain

$$\left|\frac{f(n)}{n^s}\right| < \frac{n^{\sigma-1-\varepsilon}}{n^{\sigma}} = \frac{1}{n^{1+\varepsilon}}.$$

- Hence $D(s,f)$ is absolutely convergent in the half-plane $\sigma > L$. □

Quantitaive estimates

Proposition

Let $D(s,f) = \sum_{n=1}^{\infty} f(n) n^{-s}$ be a Dirichlet series. Assume that

$$\Big| \sum_{x<n\leqslant y} f(n) \Big| \leqslant M y^{\alpha} \quad \textit{for all} \quad 0 < x < y,$$

for some $\alpha \geqslant 0$ and $M > 0$ independent of x and y.

- *Then $D(s,f)$ converges in the half-plane $\sigma > \alpha$.*
- *Furthermore, we have in this half-plane*

$$|D(s,f)| \leqslant \frac{M|s|}{\sigma - \alpha}, \quad \textit{and} \quad \Big| \sum_{x<n\leqslant y} \frac{f(n)}{n^s} \Big| \leqslant \frac{M}{x^{\sigma-\alpha}} \left(\frac{|s|}{\sigma - \alpha} + 1 \right).$$

Remark

- The latter statement ensures that $D(s,f)$ converges uniformly in any compact subset of the half plane $\sigma > \alpha$.

Proof

- Set $A(x) = \sum_{1 \leq n \leqslant x} f(n)$ and $S(x,y) = A(y) - A(x)$. By partial summation we have

$$\sum_{x<n\leqslant y} \frac{f(n)}{n^s} = \frac{S(x,y)}{y^s} + s\int_x^y \frac{S(x,u)}{u^{s+1}}du.$$

- By hypothesis we have $|S(x,y)/y^s| \leqslant My^{\alpha-\sigma}$, so that $S(x,y)/y^s$ tends to 0 as $y \to \infty$ in the half-plane $\sigma > \alpha$.
- Therefore if one of

$$D(s,f) = \sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}, \quad \text{or} \quad s\int_1^\infty \frac{A(u)}{u^{s+1}}du.$$

converges, then so does the other, and the two quantities converge to the same limit.
- But since

$$\left|\frac{A(u)}{u^{s+1}}\right| \leqslant \frac{M}{u^{\sigma-\alpha+1}},$$

we infer that the integral converges absolutely for $\sigma > \alpha$, and hence $D(s,f)$ is convergent in this half-plane.

Proof

- Therefore for all $\sigma > \alpha$, we obtain

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = s \int_1^{\infty} \frac{A(u)}{u^{s+1}} du,$$

  and hence

$$|D(s,f)| \leqslant M|s| \int_1^{\infty} \frac{du}{u^{\sigma-\alpha+1}} = \frac{M|s|}{\sigma - \alpha}.$$

- Similarly

$$\Big| \sum_{x<n\leqslant y} \frac{f(n)}{n^s} \Big| \leqslant \frac{M}{y^{\sigma-\alpha}} + M|s| \int_x^{\infty} \frac{du}{u^{\sigma-\alpha+1}} \leqslant \frac{M}{x^{\sigma-\alpha}} \left(\frac{|s|}{\sigma - \alpha} + 1 \right)$$

  as required. □

Conditional convergence of Dirichlet series

Proposition

For each Dirichlet series $D(s,f)$, there exists $\sigma_c \in \mathbb{R} \cup \{\pm\infty\}$, called the abscissa of convergence, such that $D(s,f)$ converges in the half-plane $\sigma > \sigma_c$ and does not converge in the half-plane $\sigma < \sigma_c$. Furthermore,

$$\sigma_c \leqslant \sigma_a \leqslant \sigma_c + 1.$$

Proof.

- Suppose first that $D(s,f)$ converges at a point $s_0 = \sigma_0 + it_0$ and fix a small real number $\varepsilon > 0$. By Cauchy's theorem, there exists $x_\varepsilon \geqslant 1$ such that, for all $y > x \geqslant x_\varepsilon$, we have

$$\left| \sum_{x<n\leqslant y} \frac{f(n)}{n^{s_0}} \right| \leqslant \varepsilon.$$

- Let $s = \sigma + it \in \mathbb{C}$ such that $\sigma > \sigma_0$. Using the previous proposition with s replaced by $s - s_0$ and $\alpha = 0$, we obtain

$$\left| \sum_{x<n\leqslant y} \frac{f(n)}{n^{s}} \right| \leqslant \varepsilon \left(\frac{|s-s_0|}{\sigma - \sigma_0} + 1 \right).$$

so that $D(s,f)$ converges by Cauchy's theorem.

Proof

- Now we may proceed as before. Let $S := \{s \in \mathbb{C} : D(s,f) \text{ converges}\}$.
- If $S = \varnothing$, then we put $\sigma_c = +\infty$. Otherwise define

$$\sigma_c := \inf\{\sigma : s = \sigma + it \in S\}.$$

- $D(s,f)$ does not converge if $\sigma < \sigma_a$ by the definition of σ_c.
- On the other hand, there exist points s_0 with σ_0 being arbitrarily close to σ_c at which $D(s,f)$ converges.
- By above, $D(s,f)$ converges at any point s such that $\sigma > \sigma_0$. Since σ_0 may be chosen as close to σ_c as we want, it follows that $D(s,f)$ converges at any point s such that $\sigma > \sigma_c$.
- The inequality $\sigma_c \leqslant \sigma_a \leqslant \sigma_c + 1$ remains to be shown.
- The lower bound is obvious. For the upper bound, it suffices to show that if $D(s_0,f)$ converges for some s_0, then it converges absolutely for all s such that $\sigma > \sigma_0 + 1$. Now if $D(s,f)$ converges at some point s_0, then $\lim_{n\to\infty} f(n)n^{-s_0} = 0$. Thus there exists a positive integer n_0 such that, for all $n \geqslant n_0$, we have $|f(n)| \leq n^{\sigma_0}$, hence $D(s,f)$ is absolutely convergent in the half-plane $\sigma > \sigma_0 + 1$ as required. □

Dirichlet series are holomorphic

Theorem

A Dirichlet series $D(s,f) = \sum_{n=1}^{\infty} f(n)n^{-s}$ *defines a holomorphic function of the variable s in the half-plane* $\sigma > \sigma_c$*, in which* $D(s,f)$ *can be differentiated term by term so that, for all* $s = \sigma + it$ *such that* $\sigma > \sigma_c$*, we have*

$$\partial_s^k D(s,f) = \sum_{n=1}^{\infty} \frac{(-1)^k (\log n)^k f(n)}{n^s}, \quad \textit{for} \quad k \in \mathbb{Z}_+.$$

Proof.

- We know that the partial sums of a Dirichlet series is a holomorphic function of the variable s that converges uniformly on any compact subset of the half-plane $\sigma > \sigma_c$.
- Therefore, the limit $D(s,f)$ defines a holomorphic function of the variable s in this half-plane.
- Consequently, term-by-term differentiation is allowed, and since $n^{-s} = e^{-s\log n}$, then

$$\partial_s^k n^{-s} = (-1)^k (\log n)^k n^{-s},$$

and the desired formula follows. □

Dirichlet series are determined uniquely

Proposition

Let $D(s,f) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be a Dirichlet series with abscissa of convergence σ_c. If $D(s,f) = 0$ for all s such that $\sigma > \sigma_c$, then $f(n) = 0$ for all $n \in \mathbb{Z}_+$. In particular, if $D(s,f) = D(s,f)$ for all s such that $\sigma > \sigma_c$, then $f(n) = g(n)$ for all $n \in \mathbb{Z}_+$.

Proof.

- Suppose the contrary and let $k \in \mathbb{Z}_+$ be the smallest integer such that $f(k) \neq 0$. Then $D(s,f) = \sum_{n=k}^{\infty} f(n)n^{-s} = 0$ for all s such that $\sigma > \sigma_c$. Then we have
$$G(s) = k^s D(s,f) = k^s \sum_{n=k}^{\infty} \frac{f(n)}{n^s} = 0.$$
- Therefore, for all s such that $\sigma > \sigma_c$ we have
$$G(s) = f(k) + \sum_{n=k+1}^{\infty} f(n) \left(\frac{k}{n}\right)^s = 0.$$
- Hence
$$0 = \lim_{\sigma \to \infty} G(\sigma) = f(k) \neq 0,$$
which is impossible. □

Singularity on the axis of convergence

Theorem (Landau)

If $f(n) \geqslant 0$ for all $n \in \mathbb{Z}_+$, and $D(s,f) = \sum_{n=1}^{\infty} f(n)n^{-s}$ is a Dirichlet series with abscissa of convergence $\sigma_c \in \mathbb{R}$, then $D(s,f)$ has a singularity at $s = \sigma_c$.

Proof.

- Without loss of generality, we may assume that $\sigma_c = 0$ and $|D(0,f)| < \infty$. By the Taylor expansion of $D(s,f)$ about $a > 0$ we have

$$D(s,f) = \sum_{k=0}^{\infty} \frac{(s-a)^k}{k!} \partial_s^k D(a,f) = \sum_{k=0}^{\infty} \frac{(s-a)^k}{k!} \sum_{n=1}^{\infty} \frac{(-1)^k (\log n)^k f(n)}{n^a},$$

which must converge at some $s = b < 0$. Hence

$$0 \leq \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{((a-b)\log n)^k f(n)}{n^a k!} < \infty.$$

- Each term is nonnegative, so the order of summation may be changed

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^a} \sum_{k=0}^{\infty} \frac{((a-b)\log n)^k}{k!} = \sum_{n=1}^{\infty} \frac{f(n)}{n^b},$$

which does not converge, since $b < 0 = \sigma_c$, giving a contradiction. □

Series of multiplicative functions

Theorem

Let f be a multiplicative function satisfying

$$\sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}|f(p^k)|<\infty.$$

Then the series $\sum_{n\geqslant 1} f(n)$ is absolutely convergent and we have

$$\sum_{n=1}^{\infty} f(n)=\prod_{p\in\mathbb{P}}\left(1+\sum_{k=1}^{\infty} f\left(p^k\right)\right).$$

Proof.

- Let us first notice that the inequality $\sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}|f(p^k)|<\infty$ implies the convergence of the product

$$\prod_{p\in\mathbb{P}}\left(1+\sum_{k=1}^{\infty}\left|f\left(p^k\right)\right|\right)\leq\exp\left(\sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}|f(p^k)|\right)<\infty.$$

Proof

- Now let $x \geqslant 2$ be a real number and set
$$P(x) = \prod_{p \in \mathbb{P}_{\leq x}} \left(1 + \sum_{k=1}^{\infty} |f(p^k)|\right).$$
- The convergence of the series $\sum_{k=1}^{\infty} \left| f\left(p^k\right) \right|$ enables us to rearrange the terms when we expand $P(x)$, hence
$$P(x) = \sum_{\mathrm{gpf}(n) \leq x} |f(n)|,$$
where $\mathrm{gpf}(1) = 1$ and $\mathrm{gpf}(n)$ is the greatest prime factor of $n \geq 2$.
- Since each integer $n \leqslant x$ satisfies the condition $\mathrm{gpf}(n) \leqslant x$, we have
$$\sum_{1 \leq n \leqslant x} |f(n)| \leqslant P(x)$$
- Since $P(x)$ has a finite limit as $x \to \infty$, the above inequality implies that $\sum_{n \geqslant 1} |f(n)| < \infty$. The second part of the theorem follows from the inequality
$$\left| \sum_{n=1}^{\infty} f(n) - \prod_{p \in \mathbb{P}_{\leq x}} \left(1 + \sum_{k=1}^{\infty} f(p^k)\right) \right| \leqslant \sum_{n > x} |f(n)|,$$
and the fact that the right-hand side tends to 0 as $x \to \infty$. □

Multiplication of Dirichlet series

Theorem

Let f be a multiplicative function and let $s_0 \in \mathbb{C}$. Then the three following assertions are equivalent.

(i) *One has*

$$\sum_{p \in \mathbb{P}} \sum_{k=1}^{\infty} \frac{|f(p^k)|}{p^{s_0 k}} < \infty.$$

(ii) *The series $D(s,f)$ is absolutely convergent in the half-plane $\sigma > \sigma_0$.*

(iii) *The product*

$$\prod_{p \in \mathbb{P}} \left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{sk}}\right)$$

is absolutely convergent in the half-plane $\sigma > \sigma_0$. If one of these conditions holds, then we have for all $\sigma > \sigma_0$ that

$$D(s,f) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{sk}}\right).$$

In particular, if σ_a is the abscissa of absolute convergence of $D(s,f)$, then the last identity holds for all $\sigma > \sigma_a$.