Analytic Number Theory
Lecture 8

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Jensen's formula

We denote by $D_R = \{z \in \mathbb{C} : |z| < R\}$ and $C_R = \{z \in \mathbb{C} : |z| = R\}$ the open disc and circle of radius $R \in \mathbb{R}_+$ centered at the origin.

Theorem

Let Ω be an open set that contains the closure of a disc D_R and suppose that f is holomorphic in Ω, and $f(0) \neq 0$, f vanishes nowhere on the circle C_R. If $z_1, \ldots, z_N$ denote the zeros of f inside the disc (counted with multiplicities, i.e. each zero appears in the sequence as many times as its order), then

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta. \qquad (*)$$

Proof.

The proof of the theorem consists of several steps.

- **Step 1.** First, we observe that if f_1 and f_2 are two functions satisfying the hypotheses and the conclusion of the theorem, then the product $f_1 f_2$ also satisfies the hypothesis of the theorem and formula (*). This is a simple consequence of the fact that $\log xy = \log x + \log y$ whenever $x, y \in \mathbb{R}_+$, and that the set of zeros of $f_1 f_2$ is the union of the sets of zeros of f_1 and f_2.

Proof

- **Step 2.** The function

$$g(z) = \frac{f(z)}{(z - z_1) \cdots (z - z_N)}$$

initially defined on $\Omega \setminus \{z_1, \ldots, z_N\}$, is bounded near each z_j. Therefore each z_j is a removable singularity, and hence we can write

$$f(z) = (z - z_1) \cdots (z - z_N)\, g(z),$$

where g is holomorphic in Ω and nowhere vanishing in the closure of D_R. By Step 1, it suffices to prove Jensen's formula for functions like g that vanish nowhere, and for functions of the form $z - z_j$.

- **Step 3.** We first prove (*) for a function g that vanishes nowhere in the closure of D_R. More precisely, we must establish the following identity:

$$\log|g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|g(Re^{i\theta})| d\theta. \qquad (**)$$

In a slightly larger disc, we can write $g(z) = e^{h(z)}$ where h is holomorphic in that disc. This is possible since discs are simply connected, and we can define $h = \log g$. Now $|g(z)| = |e^{h(z)}| = |e^{\mathrm{Re}(h(z)) + i\,\mathrm{Im}(h(z))}| = e^{\mathrm{Re}(h(z))}$, so that $\log|g(z)| = \mathrm{Re}(h(z))$. Then the mean value property for holomorphic functions (in our case with $h = \log g$) immediately implies the desired formula for its real part, which is precisely (**).

Proof

- **Step 4.** The last step is to prove the formula for functions of the form $f(z) = z - w$, where $w \in D_R$. That is, we must show that

$$\log|w| = \log\left(\frac{|w|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|Re^{i\theta} - w|d\theta.$$

Since $\log(|w|/R) = \log|w| - \log R$ and $\log|Re^{i\theta} - w| = \log R + \log|e^{i\theta} - w/R|$, it suffices to prove that

$$\int_0^{2\pi} \log|e^{i\theta} - a|d\theta = 0, \quad \text{whenever} \quad |a| < 1.$$

This in turn is equivalent (after the change of variables $\theta \mapsto -\theta$) to

$$\int_0^{2\pi} \log|1 - ae^{i\theta}|d\theta = 0, \quad \text{whenever} \quad |a| < 1.$$

To prove this, we use the function $F(z) = 1 - az$, which vanishes nowhere in the closure of the unit disc. As a consequence, there exists a holomorphic function G in a disc of radius greater than 1 such that $F(z) = e^{G(z)}$. Then $|F| = e^{\mathrm{Re}(G)}$, and therefore $\log|F| = \mathrm{Re}(G)$. Since $F(0) = 1$ we have $\log|F(0)| = 0$, and an application of the mean value property to the real part of holomorphic function G, which is $\log|F(z)|$ concludes the proof of the theorem. □

Growth of a holomorphic function and its number of zeros

- From Jensen's formula we can derive an identity linking the growth of a holomorphic function with its number of zeros inside a disc.
- If f is a holomorphic function on the closure of a disc D_R, we denote by $\mathfrak{n}_f(r)$ the number of zeros of f (counted with their multiplicities) inside the disc D_r, with $0 < r < R$.
- A simple but useful observation is that $\mathfrak{n}_f(r)$ is a non-decreasing function of r.

Lemma

Under the assumptions of the previous theorem, we have

$$\int_0^R \mathfrak{n}_f(r)\frac{dr}{r} = \sum_{k=1}^{N} \log\left|\frac{R}{z_k}\right|.$$

Proof.

- First we have

$$\sum_{k=1}^{N} \log\left|\frac{R}{z_k}\right| = \sum_{k=1}^{N} \int_{|z_k|}^{R} \frac{dr}{r}.$$

Proof

- If we define the characteristic function

$$\eta_k(r) = \begin{cases} 1 & \text{if } r > |z_k|, \\ 0 & \text{if } r \leq |z_k|, \end{cases}$$

  then

$$\sum_{k=1}^{N} \eta_k(r) = \mathfrak{n}_f(r).$$

- The lemma is proved using

$$\sum_{k=1}^{N} \int_{|z_k|}^{R} \frac{dr}{r} = \sum_{k=1}^{N} \int_0^R \eta_k(r) \frac{dr}{r} = \int_0^R \left(\sum_{k=1}^{N} \eta_k(r) \right) \frac{dr}{r} = \int_0^R \mathfrak{n}_f(r) \frac{dr}{r}.$$

This completes the proof of the lemma. □

Corollary

As a corollary of Jensen's formula and the previous lemma, we obtain

$$\int_0^R \mathfrak{n}_f(r) \frac{dr}{r} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \log |f(0)|.$$

Functions of finite order

- Let f be an entire function. If there exist $\rho \in \mathbb{R}_+$ and constants $A, B \in \mathbb{R}_+$ such that

$$|f(z)| \leq Ae^{B|z|^\rho} \quad \text{for all} \quad z \in \mathbb{C},$$

  then we say that f has an order of growth $\leq \rho$.
- We define the order of growth of f as

$$\rho_f = \inf \rho,$$

  where the infimum is taken over all $\rho > 0$ such that f has an order of growth $\leq \rho$.
- For example, the order of growth of the function e^{z^2} is 2.

Theorem

If f is an entire function that has an order of growth $\leq \rho$, then:

(i) $\mathfrak{n}_f(r) \leq Cr^\rho$ *for some* $C > 0$ *and all sufficiently large* r.

(ii) *If* $z_1, z_2, \ldots$ *denote the zeros of* f, *with* $z_k \neq 0$, *then for all* $s > \rho$ *we have*

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty.$$

Proof

- It suffices to prove the estimate for $\mathfrak{n}_f(r)$ when $f(0) \neq 0$. Indeed, consider $F(z) = f(z)/z^l$, where l is the order of the zero of f at the origin. Then $\mathfrak{n}_f(r)$ and $\mathfrak{n}_F(r)$ differ only by a constant, and F also has an of order of growth $\leq \rho$.
- If $f(0) \neq 0$ we may use formula from the previous corollary, namely

$$\int_0^R \mathfrak{n}_f(x) \frac{dx}{x} = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \log|f(0)|.$$

- Choosing $R = 2r$, this formula implies

$$\int_r^{2r} \mathfrak{n}_f(x) \frac{dx}{x} \leq \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \log|f(0)|.$$

- On the one hand, since $\mathfrak{n}_f(r)$ is increasing, we have

$$\int_r^{2r} \mathfrak{n}_f(x) \frac{dx}{x} \geq \mathfrak{n}_f(r) \int_r^{2r} \frac{dx}{x} = \mathfrak{n}_f(r)[\log 2r - \log r] = \mathfrak{n}_f(r) \log 2.$$

- On the other hand, the growth condition on f (for all large r) gives

$$\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta \leq \int_0^{2\pi} \log|Ae^{BR^\rho}| d\theta \leq C' r^\rho.$$

- Consequently, $\mathfrak{n}_f(r) \leq C r^\rho$ for an appropriate $C > 0$ and all sufficiently large r.

Proof

- The following estimates prove the second part of the theorem:

$$\begin{aligned}\sum_{|z_k|\geq 1} |z_k|^{-s} &= \sum_{j=0}^{\infty}\left(\sum_{2^j\leq |z_k|<2^{j+1}} |z_k|^{-s}\right)\\ &\leq \sum_{j=0}^{\infty} 2^{-js}\mathfrak{n}\left(2^{j+1}\right)\\ &\leq c\sum_{j=0}^{\infty} 2^{-js}2^{(j+1)\rho}\\ &\leq c'\sum_{j=0}^{\infty}\left(2^{\rho-s}\right)^j < \infty\end{aligned}$$

- The last series converges because $s > \rho$.

This completes the proof of the theorem. □

Infinite products

- Given a sequence $(a_n)_{n\in\mathbb{Z}_+} \subseteq \mathbb{C}$, we say that the product

$$\prod_{n=1}^{\infty} (1 + a_n)$$

  converges if the limit

$$\lim_{N\to\infty} \prod_{n=1}^{N} (1 + a_n)$$

  of the partial products exists.
- A useful necessary condition that guarantees the existence of a product is contained in the following proposition.

Proposition

If $\sum_{n\in\mathbb{Z}_+} |a_n| < \infty$, then the product

$$\prod_{n=1}^{\infty} (1 + a_n)$$

converges. Moreover, the product converges to 0 if and only if one of its factors is 0.

Proof

- If $\sum_{n\in\mathbb{Z}_+} |a_n|$ converges, then for all large n we must have $|a_n| < 1/2$. We may assume, without loss of generality, that this inequality holds for all $n \in \mathbb{Z}_+$.
- Hence, we can define $\log(1 + a_n)$ by the usual power series, and this logarithm satisfies the property that $1 + z = e^{\log(1+z)}$ whenever $|z| < 1$.
- Hence we may write the partial products as follows:

$$\prod_{n=1}^{N} (1 + a_n) = \prod_{n=1}^{N} e^{\log(1+a_n)} = e^{B_N}$$

  where $B_N = \sum_{n=1}^{N} b_n$ with $b_n = \log(1 + a_n)$.
- By the power series expansion we see that $|\log(1+z)| \le 2|z|$, if $|z| < 1/2$. Hence $|b_n| \le 2\,|a_n|$, so B_N converges as $N \to \infty$ to a complex number, say B.
- Since the exponential function is continuous, we conclude that e^{B_N} converges to e^B as $N \to \infty$, and the first part follows.
- Observe also that if $1 + a_n \neq 0$ for all $n \in \mathbb{Z}_+$, then the product converges to a non-zero limit since it is expressed as e^B. □

Infinite products of holomorphic functions

Proposition

Suppose $(F_n)_{n\in\mathbb{Z}_+}$ *is a sequence of holomorphic functions on the open set* Ω*. If there exist constants* $c_n > 0$ *such that*

$$\sum_{n\in\mathbb{Z}_+} c_n < \infty \quad \text{and} \quad |F_n(z) - 1| \le c_n \quad \text{for all} \quad z \in \Omega,$$

then:

(i) *The product* $\prod_{n=1}^{\infty} F_n(z)$ *converges uniformly in* Ω *to a holomorphic function* $F(z)$*.*

(ii) *If* $F_n(z)$ *does not vanish for any* n*, then*

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}.$$

Proof

- To prove the first statement, note that for each z we may argue as in the previous proposition if we write $F_n(z) = 1 + a_n(z)$, with $|a_n(z)| \leq c_n$.
- Then, we observe that the estimates are actually uniform in z because the c_n 's are constants. It follows that the product converges uniformly to a holomorphic function, which we denote by $F(z)$.
- To establish the second part of the theorem, suppose that K is a compact subset of Ω, and let

$$G_N(z) = \prod_{n=1}^{N} F_n(z).$$

- We have just proved that $\lim_{N\to\infty} G_N = F$ uniformly in Ω. Hence, the sequence $(G'_N)_{N\in\mathbb{Z}_+}$ converges uniformly to F' in K.
- Since G_N is uniformly bounded from below on K, we conclude that $\lim_{N\to\infty} G'_N/G_N = F'/F$ uniformly on K, and because K is an arbitrary compact subset of Ω, the limit holds for every point of Ω.
- Moreover, a simple calculation yields

$$\frac{G'_N}{G_N} = \sum_{n=1}^{N} \frac{F'_n}{F_n},$$

so part (ii) of the proposition is also proved. □

Canonical factors

- For each integer $k \geq 0$ we define canonical factors by
$$E_0(z) = 1 - z \quad \text{and} \quad E_k(z) = (1-z)e^{z+z^2/2+\cdots+z^k/k}, \quad \text{for } k \geq 1.$$
The integer k is called the degree of the canonical factor.

Lemma

If $|z| \leq 1/2$, *then* $|E_k(z) - 1| \leq 2e|z|^{k+1}$.

Proof.

- If $|z| \leq 1/2$, then with the logarithm defined in terms of the power series, we have $1 - z = e^{\log(1-z)}$, and therefore
$$E_k(z) = e^{\log(1-z)+z+z^2/2+\cdots+z^k/k} = e^w$$
where $w = -\sum_{n=k+1}^{\infty} z^n/n$. Observe that since $|z| \leq 1/2$ we have
$$|w| \leq |z|^{k+1} \sum_{n=k+1}^{\infty} |z|^{n-k-1}/n \leq |z|^{k+1} \sum_{j=0}^{\infty} 2^{-j} \leq 2|z|^{k+1}.$$
- In particular, we have $|w| \leq 1$ and this implies that
$$|1 - E_k(z)| = |1 - e^w| \leq e|w| \leq 2e|z|^{k+1}. \qquad \square$$

Weierstrass infinite products

Theorem

Given any sequence $(a_n)_{n\in\mathbb{Z}_+} \subseteq \mathbb{C}$ with $\lim_{n\to\infty} |a_n| = \infty$, there exists an entire function f that vanishes at all $z = a_n$ and nowhere else. Any other such entire function is of the form $f(z)e^{g(z)}$, where g is entire.

Proof.

- Recall that if a holomorphic function f vanishes at $z = a$, then the multiplicity of the zero a is the integer m so that

$$f(z) = (z-a)^m g(z),$$

  where g is holomorphic and nowhere vanishing in a neighborhood of a.
- To begin the proof, note first that if f_1 and f_2 are two entire functions that vanish at all $z = a_n$ and nowhere else, then f_1/f_2 has removable singularities at all the points a_n. Hence f_1/f_2 is entire and vanishes nowhere, so that there exists an entire function g with

$$f_1(z)/f_2(z) = e^{g(z)}.$$

- Therefore $f_1(z) = f_2(z)e^{g(z)}$ as desired.

Proof

- We have to construct a function that vanishes at all the points of the sequence $(a_n)_{n \in \mathbb{Z}_+}$ and nowhere else.
- Suppose that we are given a zero of order m at the origin, and that $a_1, a_2 \ldots$ are all non-zero. Then we define the Weierstrass product by

$$f(z) = z^m \prod_{n=1}^{\infty} E_n\left(z/a_n\right).$$

- We claim that this function has the required properties; that is,
  (i) f is entire with a zero of order m at the origin;
  (ii) f has zeros at each point of the sequence $(a_n)_{n \in \mathbb{Z}_+}$;
  (iii) f vanishes nowhere else.
- Fix $R > 0$, and suppose that z belongs to the disc $|z| < R$. We shall prove that f has all the desired properties in this disc, and since R is arbitrary, this will prove the theorem.

Proof

- We can consider two types of factors in the formula defining f, with the choice depending on whether $|a_n| \leq 2R$ or $|a_n| > 2R$.
- There are only finitely many terms of the first kind (since $\lim_{n\to\infty} |a_n| = \infty$), and we see that the finite product vanishes at all $z = a_n$ with $|a_n| < R$.
- If $|a_n| \geq 2R$, we have $|z/a_n| \leq 1/2$, hence the previous lemma implies

$$|E_n(z/a_n) - 1| \leq 2e \left|\frac{z}{a_n}\right|^{n+1} \leq \frac{e}{2^n}.$$

- Therefore, the product

$$\prod_{|a_n|\geq 2R} E_n(z/a_n)$$

  defines a holomorphic function when $|z| < R$, and does not vanish in that disc by the previous propositions.
- This shows that the function f has the desired properties, and the proof of Weierstrass's theorem is complete. □

Hadamard's theorem

Theorem (Hadamard)

Suppose f is entire and has growth order ρ_0. Let $k \in \mathbb{Z}$ be so that $k \leq \rho_0 < k+1$. If $a_1, a_2, \ldots$ denote the (non-zero) zeros of f, then

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(z/a_n\right),$$

where P is a polynomial of degree $\leq k$, and m is the order of the zero of f at $z = 0$, and E_k are the canonical factors for $k \in \mathbb{N}$.

- We gather a few lemmas needed in the proof of Hadamard's theorem.

Lemma

The canonical products satisfy

$$|E_k(z)| \geq e^{-c|z|^{k+1}} \quad \text{if} \quad |z| \leq 1/2,$$

and

$$|E_k(z)| \geq |1-z| e^{-c|z|^k} \quad \text{if} \quad |z| \geq 1/2.$$

Here, we allow the implied constant $c = c_k$ to depend on $k \in \mathbb{N}$.

Proof

- If $|z| \leq 1/2$ we can use the power series to define the logarithm of $1 - z$, so that
$$E_k(z) = e^{\log(1-z)+\sum_{n=1}^{k} z^n/n} = e^{-\sum_{n=k+1}^{\infty} z^n/n} = e^w.$$
Since $|e^w| \geq e^{-|w|}$ and $|w| \leq c|z|^{k+1}$, the first part follows.
- For the second part, simply observe that if $|z| \geq 1/2$, then
$$|E_k(z)| = |1 - z| \left| e^{z+z^2/2+\cdots+z^k/k} \right|,$$
and that there exists $c' > 0$ such that
$$\left| e^{z+z^2/2+\cdots+z^k/k} \right| \geq e^{-\left|z+z^2/2+\cdots+z^k/k\right|} \geq e^{-c'|z|^k}. \qquad \square$$

Lemma

For any $s \in \mathbb{R}_+$ with $\rho_0 < s < k + 1$, we have
$$\left| \prod_{n=1}^{\infty} E_k(z/a_n) \right| \geq e^{-c|z|^s},$$
except possibly when z belongs to the union of the discs centered at a_n of radius $|a_n|^{-k-1}$, for $n \in \mathbb{Z}_+$.

Proof

- First, we write

$$\prod_{n=1}^{\infty} E_k(z/a_n) = \prod_{|a_n|\le 2|z|} E_k(z/a_n) \prod_{|a_n|>2|z|} E_k(z/a_n)$$

- For the second product the estimate asserted above holds for all $z \in \mathbb{C}$. Indeed, by the previous lemma

$$\left| \prod_{|a_n|>2|z|} E_k(z/a_n) \right| = \prod_{|a_n|>2|z|} |E_k(z/a_n)|$$
$$\ge \prod_{|a_n|>2|z|} e^{-c|z/a_n|^{k+1}} \ge e^{-c|z|^{k+1}\sum_{|a_n|>2|z|}|a_n|^{-k-1}}.$$

- But $|a_n| > 2|z|$ and $s < k+1$, so we must have

$$|a_n|^{-k-1} = |a_n|^{-s} |a_n|^{s-k-1} \le C |a_n|^{-s} |z|^{s-k-1}.$$

- Therefore, the fact that $\sum_{n\in\mathbb{Z}_+} |a_n|^{-s}$ converges implies that

$$\left| \prod_{|a_n|>2|z|} E_k(z/a_n) \right| \ge e^{-c|z|^s}$$

for some constant $c > 0$, which may depend on k and s.

Proof

- To estimate the first product, we use the second part of the previous lemma, and write

$$\left| \prod_{|a_n| \le 2|z|} E_k\left(z/a_n\right) \right| \ge \prod_{|a_n| \le 2|z|} \left| 1 - \frac{z}{a_n} \right| \prod_{|a_n| \le 2|z|} e^{-c|z/a_n|^k}. \qquad (*)$$

- We now note that

$$\prod_{|a_n| \le 2|z|} e^{-c|z/a_n|^k} = e^{-c|z|^k \sum_{|a_n| \le 2|z|} |a_n|^{-k}},$$

and again, we have $|a_n|^{-k} = |a_n|^{-s} |a_n|^{s-k} \le C |a_n|^{-s} |z|^{s-k}$, thereby proving that

$$\prod_{|a_n| \le 2|z|} e^{-c'|z/a_n|^k} \ge e^{-c|z|^s}.$$

- The estimate on the first product on the right-hand side of (*) will require the restriction on z imposed in the statement of the lemma.
- Indeed, whenever z does not belong to a disc of radius $|a_n|^{-k-1}$ centered at a_n, we must have $|a_n - z| \ge |a_n|^{-k-1}$.

Proof

- Therefore

$$\prod_{|a_n|\le 2|z|}\left|1-\frac{z}{a_n}\right| = \prod_{|a_n|\le 2|z|}\left|\frac{a_n-z}{a_n}\right|$$
$$\ge \prod_{|a_n|\le 2|z|}|a_n|^{-k-1}\,|a_n|^{-1}$$
$$= \prod_{|a_n|\le 2|z|}|a_n|^{-k-2}\,.$$

- Finally, the estimate for the first product follows from the fact that

$$(k+2)\sum_{|a_n|\le 2|z|}\log|a_n| \le (k+2)\mathfrak{n}_f(2|z|)\log 2|z|$$
$$\le c|z|^s\log 2|z|$$
$$\le c'|z|^{s'}$$

for any $s' > s$, and the second inequality follows as $\mathfrak{n}(2|z|) \le c|z|^s$.
- Since we restricted s to satisfy $s > \rho_0$, we can take an initial s sufficiently close to ρ_0, so that the assertion of the lemma is established (with s being replaced by s'). □

Useful corollary

Corollary

There exists a sequence of radii, $r_1, r_2, \ldots$, with $r_m \to \infty$, such that

$$\left| \prod_{n=1}^{\infty} E_k\left(z/a_n\right) \right| \geq e^{-c|z|^s} \quad \textit{for} \quad |z| = r_m.$$

Proof.

- Since $\sum_{n \in \mathbb{Z}_+} |a_n|^{-k-1} < \infty$, there exists $N \in \mathbb{Z}_+$ so that
$$\sum_{n=N}^{\infty} |a_n|^{-k-1} < 1/10.$$
- Therefore, given any two consecutive large integers L and $L+1$, we can find $r \in \mathbb{R}_+$ with $L \leq r \leq L+1$, such that the circle of radius r centered at the origin does not intersect the forbidden discs from the previous lemma.
- For otherwise, the union of the intervals
$$I_n = \left[|a_n| - \frac{1}{|a_n|^{k+1}}, |a_n| + \frac{1}{|a_n|^{k+1}} \right]$$
(which are of length $2|a_n|^{-k-1}$) would cover all the interval $[L, L+1]$.
- This would imply $2\sum_{n=N}^{\infty} |a_n|^{-k-1} \geq 1$, which is a contradiction. We can then apply the previous lemma with $|z| = r$ to conclude the proof. □

Proof of Hadamard's theorem

- Let
$$E(z) = z^m \prod_{n=1}^{\infty} E_k\left(z/a_n\right).$$
- To prove that E is entire, we repeat the argument in the proof of Weierstrass theorem. Namely, we have
$$\left|E_k\left(z/a_n\right) - 1\right| \leq 2e\left|\frac{z}{a_n}\right|^{k+1}, \quad \text{for all large} \quad n \in \mathbb{Z}_+,$$
and that the series $\sum_{n\in\mathbb{Z}_+} |a_n|^{-k-1}$ converges. (Recall $\rho_0 < s < k+1$.)
- Moreover, E has the zeros of f, therefore f/E is holomorphic and nowhere vanishing. Hence
$$\frac{f(z)}{E(z)} = e^{g(z)}$$
for some entire function g.
- By the fact that f has growth order ρ_0, and because of the estimate from below for E obtained in the previous corollary, we have
$$e^{\mathrm{Re}(g(z))} = \left|\frac{f(z)}{E(z)}\right| \leq c' e^{c|z|^s}, \quad \text{whenever} \quad |z| = r_m.$$

Proof of Hadamard's theorem

- This proves that

$$\operatorname{Re}(g(z)) \leq C|z|^s, \quad \text{for} \quad |z| = r_m,$$

  where $(r_m)_{m\in\mathbb{Z}_+} \subseteq \mathbb{R}_+$ is a a sequence such that $\lim_{m\to\infty} r_m = \infty$.
- We have to prove that g is a polynomial of degree $\leq s$.
- We can expand g in a power series centered at the origin

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

- As a simple application of Cauchy's integral formulas, we may write

$$\frac{1}{2\pi}\int_0^{2\pi} g\left(re^{i\theta}\right) e^{-in\theta} d\theta = \begin{cases} a_n r^n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

- By taking complex conjugates we find that

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{g\left(re^{i\theta}\right)} e^{-in\theta} d\theta = 0$$

  whenever $n > 0$.

Proof

- Since $2u = g + \overline{g}$ we add the above two equations and obtain

$$a_n r^n = \frac{1}{\pi} \int_0^{2\pi} u\left(re^{i\theta}\right) e^{-in\theta} d\theta, \quad \text{whenever} \quad n > 0.$$

- For $n = 0$ we find that

$$2\,\mathrm{Re}\left(a_0\right) = \frac{1}{\pi} \int_0^{2\pi} u\left(re^{i\theta}\right) d\theta.$$

- Now we recall the simple fact that whenever $n \neq 0$, the integral of $e^{-in\theta}$ over any circle centered at the origin vanishes. Therefore

$$a_n = \frac{1}{\pi r^n} \int_0^{2\pi} \left[u\left(re^{i\theta}\right) - Cr^s\right] e^{-in\theta} d\theta \quad \text{when} \quad n > 0.$$

- Taking $r = r_m$, we consequently obtain

$$|a_n| \leq \frac{1}{\pi r_m^n} \int_0^{2\pi} \left[Cr_m^s - u\left(r_m e^{i\theta}\right)\right] d\theta \leq 2Cr_m^{s-n} - 2\,\mathrm{Re}\left(a_0\right) r_m^{-n}.$$

- Letting $m \to \infty$ we deduce $a_n = 0$ for any $n > s$. This completes the proof of Hadamard's theorem. □

Example

- The function $\sin \pi s$ is entire and of order one, and its zeros are at $s = 0, \pm 1, \pm 2, \ldots$, and so, by Hadamard's theorem we can write

$$\sin \pi s = s e^{H(s)} \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right),$$

  where $H(s) = as + b$.
- Taking the logarithmic derivative of this equation, we find that

$$\pi \frac{\cos \pi s}{\sin \pi s} = \frac{1}{s} + H'(s) - \sum_{n=1}^{\infty} \frac{2s}{n^2 - s^2}.$$

- Passage to the limit as $s \to 0$ gives $a = 0$, and so $H(s) = b$. Thus,

$$\frac{\sin \pi s}{s} = c \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right).$$

- Passing again to the limit as $s \to 0$ gives $c = \pi$, i.e.

$$\sin \pi s = \pi s \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right).$$

Euler's gamma function

- The Euler gamma function $\Gamma(s)$ is defined by the equation

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$$

  where γ is Euler's constant.
- It follows from the definition that $\Gamma^{-1}(s)$ is an entire function of order at most one.
- Moreover, $\Gamma(s)$ is an analytic function in the entire s-plane except for the points $s = 0, -1, -2, \ldots$, where it has simple poles.

Theorem (Euler's formula)

For every $s \in \mathbb{C} \setminus \{-n : n \in \mathbb{N}\}$, we have

$$\Gamma(s) = \frac{1}{s} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{s} \left(1 + \frac{s}{n}\right)^{-1}.$$

In other words, $\Gamma(s)$ is a meromorphic function on $\mathbb{C}$ with simple poles at 0 and at the negative integers and with no zeros.

Proof

- From the definition of an infinite product and from the definition of the function $\Gamma(s)$, we obtain

$$
\begin{aligned}
\frac{1}{\Gamma(s)} &= s \lim_{m\to\infty} e^{s\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\log m\right)} \cdot \lim_{m\to\infty} \prod_{n=1}^{m} \left(1+\frac{s}{n}\right) e^{-\frac{s}{n}} \\
&= s \lim_{m\to\infty} m^{-s} \prod_{n=1}^{m} \left(1+\frac{s}{n}\right) \\
&= s \lim_{m\to\infty} \prod_{n=1}^{m-1} \left(1+\frac{1}{n}\right)^{-s} \prod_{n=1}^{m} \left(1+\frac{s}{n}\right) \\
&= s \lim_{m\to\infty} \prod_{n=1}^{m} \left(1+\frac{1}{n}\right)^{-s} \left(1+\frac{s}{n}\right) \left(1+\frac{1}{m}\right)^{s} \\
&= s \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-s} \left(1+\frac{s}{n}\right),
\end{aligned}
$$

which is what we had to prove. □

Properties of Gamma function

Corollary

For every $s \in \mathbb{C} \setminus \{-n : n \in \mathbb{N}\}$, we have

$$\Gamma(s) = \lim_{n\to\infty} \frac{(n-1)! \cdot n^s}{s(s+1) \cdot \ldots \cdot (s+n-1)}.$$

Proof.

- *From the previous theorem we have*

$$\begin{aligned}\Gamma(s) &= \lim_{n\to\infty} s^{-1} \prod_{m=1}^{n-1} \left(1 + \frac{1}{m}\right)^s \left(1 + \frac{s}{m}\right)^{-1} \\ &= \lim_{n\to\infty} \frac{2^s \cdot \frac{3^s}{2^s} \cdot \ldots \cdot \frac{n^s}{(n-1)^s}}{s \cdot \frac{(s+1)}{1} \cdot \ldots \cdot \frac{(s+n-1)}{n-1}} \\ &= \lim_{n\to\infty} \frac{1 \cdot 2 \cdot \ldots \cdot (n-1) n^s}{s \cdot (s+1) \cdot \ldots \cdot (s+n-1)}\end{aligned}$$

as desired. □

Corollary

We also have $\Gamma(1) = \Gamma(2) = 1$.

Properties of Gamma function

Theorem (Functional equation)

- *We have* $\Gamma(s+1) = s\Gamma(s)$ *for all* $s \in \mathbb{C} \setminus \{-n : n \in \mathbb{N}\}$.
- *In particular,* $\Gamma(n+1) = n!$ *for all* $n \in \mathbb{N}$*, and* $\mathrm{res}_{s=-m}\Gamma(s) = \frac{(-1)^m}{m!}$.

Proof.

- We have

$$\begin{aligned}\frac{\Gamma(s+1)}{\Gamma(s)} &= \frac{s}{s+1} \lim_{m\to\infty} \prod_{n=1}^{m} \frac{\left(1+\frac{1}{n}\right)^{s+1}\left(1+\frac{s+1}{n}\right)^{-1}}{\left(1+\frac{1}{n}\right)^{s}\left(1+\frac{s}{n}\right)^{-1}} \\ &= \frac{s}{s+1} \lim_{m\to\infty} \prod_{n=1}^{m} \frac{n+1}{n} \cdot \frac{n+s}{n+s+1} \\ &= \frac{s}{s+1} \lim_{m\to\infty} \frac{(m+1)(s+1)}{m+1+s} = s.\end{aligned}$$

This completes the proof. □

Corollary (Duplication formula)

$$\Gamma(2s)\Gamma(1/2) = 2^{2s-1}\Gamma(s)\Gamma(s+1/2) \quad \textit{for all} \quad s \in \mathbb{C} \setminus (-\mathbb{N}).$$

Properties of Gamma function

Theorem (Reflection formula)

$$\frac{\sin \pi s}{\pi} = \frac{1}{\Gamma(s)\Gamma(1-s)} \quad \textit{for all} \quad s \in \mathbb{C}.$$

Proof.

- We know that
$$\frac{\sin \pi s}{\pi s} = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right).$$
- On the other hand, we have
$$\frac{1}{\Gamma(s)\Gamma(-s)} = -s^2 \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right)$$
- But we also know that $\Gamma(1-s) = -s\Gamma(-s)$, and the result follows.

□

Corollary

As a corollary we obtain that $\Gamma(1/2) = \sqrt{\pi}$.

Integral representation of the gamma function

Theorem (Integral representation)

Suppose that $\mathrm{Re}(s) > 0$. *Then*

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$$

Proof.

- We know that

$$\Gamma(s) = \lim_{n\to\infty} \frac{n! \cdot n^s}{s(s+1)(s+2)\cdots(s+n)}.$$

- We have to establish two things. Firstly, we will show that

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{s-1} dt = \frac{n! \cdot n^s}{s(s+1)(s+2)\cdot \ldots \cdot (s+n)} \quad \text{for all} \quad n \in \mathbb{Z}_+.$$

- Secondly, we will show that

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{s-1} dt = \int_0^\infty e^{-t} t^{s-1} dt,$$

which will complete the proof.

Proof

- Indeed, when $s > 0$ the above integral converges and we have

$$\begin{aligned}
\int_0^n \left(1 - \frac{t}{n}\right)^n t^{s-1} dt &= n^s \int_0^1 (1-u)^n u^{s-1} du \\
&= n^s \frac{n}{s} \int_0^1 (1-u)^{n-1} u^s du \\
&= n^s \frac{n(n-1)}{s(s+1)} \int_0^1 (1-u)^{n-2} u^{s+1} du \\
&\vdots \\
&= n^s \frac{n(n-1)\cdot \ldots \cdot 1}{s(s+1)\cdot \ldots \cdot (s+n-1)} \int_0^1 u^{s+n-1} du \\
&= \frac{n! \cdot n^s}{s(s+1)(s+2)\cdot \ldots \cdot (s+n)}.
\end{aligned}$$

- Thus, it suffices to prove that

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{s-1} dt = \int_0^\infty e^{-t} t^{s-1} dt.$$

Proof

- To this end, we consider the functions
$$f_n(t) = \begin{cases} (1 - t/n)^n t^{s-1} & \text{if } 0 \le t \le n, \\ 0 & \text{if } t > n. \end{cases}$$
- Each of these functions is in $L^1([0,\infty))$ and satisfies the inequality
$$|f_n(t)| \le e^{-t} t^{\sigma - 1}, \quad \text{where} \quad \sigma = \mathrm{Re}(s).$$
- The last inequality is easily verified by taking logarithms and noting
$$n \log\left(1 - \frac{t}{n}\right) = -t - \frac{t^2}{2n} - \frac{t^3}{3n^2} - \cdots < -t.$$
- Furthermore,
$$\lim_{n\to\infty} f_n(t) = t^{s-1} \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n = e^{-t} t^{s-1}.$$
- Since the function $e^{-t} t^{\sigma-1}$ is in $L^1([0,\infty))$, the dominated convergence theorem yields
$$\lim_{n\to\infty} \int_0^\infty f_n(t) dt = \int_0^\infty \lim_{n\to\infty} f_n(t) dt = \int_0^\infty e^{-t} t^{s-1} dt,$$
which completes the proof of the lemma. □

Stirling's formula

Theorem (Stirling's formula. Exercise)

Suppose that $s \in$ *mathbbC such that* $|\arg s| < \pi$. *Then*

$$\log \Gamma(s) = (s - 1/2) \log s - s + \log \sqrt{2\pi} + \int_0^\infty \frac{\psi(u)}{u+s} du.$$

Here $\log s$ *denotes the principal branch of the logarithm and* $\psi(u) = \{u\} - 1/2$.

Corollary (Exercise)

Suppose that $0 < \delta < \pi$ *and* $|\arg s| < \pi - \delta$. *Then*

$$\log \Gamma(s) = (s - 1/2) \log s - s + \log \sqrt{2\pi} + O\left(|s|^{-1}\right)$$

uniformly as $|s| \to \infty$, *and* $\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O\left(|s|^{-1}\right)$, *where the implied constants depending at most on* δ.

Corollary (Exercise)

Suppose that $\alpha \leq \sigma \leq \beta$ *and* $|t| \geq 1$. *Then*

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}|t|^{\sigma - 1/2} \exp(-\pi|t|/2)\left(1 + O(|t|^{-1})\right),$$

where the implied constant depending at most on α *and* β.

Riemann zeta-function

Definition (Riemann zeta-function)

The Riemann zeta-function $\zeta(s)$ is defined for all complex numbers $s = \sigma + it$ such that $\sigma > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

- By the absolute convergence all complex numbers $s = \sigma + it$ such that $\sigma > 1$ we also have the Euler product formula

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

- The Euler product formula enables us to see that $\zeta(s) \neq 0$ in the half-plane $\sigma > 1$. Indeed, for $\sigma > 1$ we have

$$\frac{1}{|\zeta(s)|} = \prod_{p \in \mathbb{P}} \left|1 - \frac{1}{p^s}\right| \leq \prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^\sigma}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq 1 + \int_1^{\infty} \frac{dt}{t^\sigma} = \frac{\sigma}{\sigma - 1}.$$

  Thus $|\zeta(s)| \geq \frac{\sigma - 1}{\sigma} > 0$.

Remarks

Euler's summation formula

If $f \in C^1([a,b])$, and $\psi(x) = \{x\} - 1/2$ for $x \in \mathbb{R}$, then by summation by parts we obtain the following identity

$$\sum_{a<n\leqslant b} f(n) = \int_a^b f(x)dx + f(a)\psi(a) - f(b)\psi(b) + \int_a^b f'(x)\psi(x)dx.$$

- By the summation by parts formula we can derive
- Let $x \geqslant 1$ be a real number and $s = \sigma + it$ with $\sigma > 1$. By the Euler summation formula with $a = 1, b = x$ and $f(x) = x^{-s}$, we can write

$$\sum_{n\leqslant x} \frac{1}{n^s} = \frac{1}{2} + \frac{1-x^{1-s}}{s-1} - \frac{\psi(x)}{x^s} - s\int_1^x \frac{\psi(u)}{u^{s+1}}du.$$

- Taking $x \to \infty$ we obtain

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} - s\int_1^\infty \frac{\psi(u)}{u^{s+1}}du. \qquad (*)$$

- Since $|\psi(x)| \leqslant \frac{1}{2}$, the integral converges for $\sigma > 0$ and is uniformly convergent in any finite region to the right of the line $\sigma = 0$.
- This implies that it defines an analytic function in the half-plane $\sigma > 0$, and therefore (*) extends ζ to a meromorphic function in this half-plane, which is analytic except for a simple pole at $s = 1$ with residue 1.

The Theta function

- Replacing x by $\pi n^2 x$ in the integral defining $\Gamma(s/2)$ gives

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^{\infty} x^{s/2-1} e^{-\pi n^2 x} dx \quad \text{for all} \quad \sigma > 0.$$

- The purpose is to sum both sides of this equation. To this end, we define the following two Theta functions. For all $x > 0$, we set

$$\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x} \quad \text{and} \quad \theta(x) = 2\omega(x) + 1 = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}.$$

- Then $g(t) = e^{-\pi t^2}$ satisfies $\int_{\mathbb{R}} g(t)dt = 1$, and its Fourier transform is

$$\widehat{g}(u) = e^{-\pi u^2}.$$

- For a Schwartz function f, by the Poisson summation formula, we have $\sum_{n\in\mathbb{Z}} \widehat{f}(n) = \sum_{n\in\mathbb{Z}} f(n)$, hence

$$\theta(x) = \sum_{n\in\mathbb{Z}} g(\sqrt{x}n) = x^{-1/2}\theta(x^{-1}) \quad \text{for all} \quad x > 0.$$

The Theta function

- Summing this equation over $n \in \mathbb{Z}_+$ and interchanging the sum and integral, we obtain for all $\sigma > 1$ that

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{s/2-1}\omega(x)dx,$$

since the sum and integral converge absolutely in the half-plane $\sigma > 1$.

- Splitting the integral $\int_0^\infty = \int_0^1 + \int_1^\infty$ and changing the variables $x \mapsto 1/x$ in the first integral yields

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^\infty x^{s/2-1}\omega(x)dx + \int_1^\infty x^{-s/2-1}\omega\left(\frac{1}{x}\right)dx.$$

- Using $\theta(x^{-1}) = x^{1/2}\theta(x)$ we may write

$$\omega\left(\frac{1}{x}\right) = x^{1/2}\omega(x) + \frac{x^{1/2}-1}{2},$$

and consequently we obtain

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \omega(x)\left(x^{s/2} + x^{(1-s)/2}\right)\frac{dx}{x},$$

whenever $\sigma > 1$.

Functional equation

Theorem

Let

$$\begin{aligned}\Xi(s) &= \pi^{-s/2}\Gamma(s/2)\zeta(s)\\ &= -\frac{1}{s} + \frac{1}{s-1} + \frac{1}{2}\int_1^\infty (\theta(x)-1)\left(x^{s/2} + x^{(1-s)/2}\right)\frac{dx}{x},\end{aligned}$$

where θ is the Theta function

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}.$$

- *Then the function $\Xi(s)$ can be extended analytically in the whole complex plane to a meromorphic function having simple poles at $s = 0$ and $s = 1$, and satisfies the functional equation $\Xi(s) = \Xi(1-s)$.*
- *Thus the Riemann zeta-function can be extended analytically in the whole complex plane to a meromorphic function having a simple pole at $s = 1$ with residue 1. Furthermore, for all $s \in \mathbb{C}\backslash\{1\}$, we have*

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

Proof

- For $\sigma > 1$ we have

$$\Xi(s) = -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \omega(x) \left(x^{s/2} + x^{(1-s)/2} \right) \frac{dx}{x}. \qquad (*)$$

- Since $\omega(x) = O(e^{-\pi x})$ as $x \to \infty$, we infer that the integral is absolutely convergent for all $s \in \mathbb{C}$ whereas the left-hand side is a meromorphic function on $\sigma > 0$. This implies that
  (i) The identity (*) is valid for all $\sigma > 0$.
  (ii) The function $\Xi(s)$ can be defined by this identity as a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0$ and $s = 1$.
  (iii) Since the right-hand side of (*) is invariant under the substitution $s \mapsto 1 - s$, we obtain $\Xi(s) = \Xi(1 - s)$.
  (iv) The function $s \mapsto \xi(s) := s(s-1)\Xi(s)$ is entire on $\mathbb{C}$. Indeed, if $\sigma > 0$, the factor $s - 1$ counters the pole at $s = 1$, and the result on all $\mathbb{C}$ follows from the functional equation.
- It remains to show that the functional equation can be written as

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

Proof

- Since $\Xi(s) = \Xi(1-s)$, we have

$$\Gamma(s/2)\zeta(s) = \pi^{s/2}\Xi(s) = \pi^{s/2}\Xi(1-s) = \pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

- Multiplying both sides by $\pi^{-1/2}2^{s-1}\Gamma\left(\frac{1+s}{2}\right)$ and using the duplication formula, asserting that $\Gamma(s) = \pi^{-1/2}2^{s-1}\Gamma\left(s/2\right)\Gamma\left((s+1)/2\right)$ we see

$$\Gamma(s)\zeta(s) = (2\pi)^{s-1}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\zeta(1-s)$$

- Now the reflection formula $\frac{\sin \pi s}{\pi} = \frac{1}{\Gamma(s)\Gamma(1-s)}$, implies that

$$\zeta(s) = (2\pi)^{s-1}\left(\frac{\sin \pi s}{\sin(\pi(1+s)/2)}\right)\Gamma(1-s)\zeta(1-s)$$

  and the result follows from the identity

$$\sin \pi s = 2\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi}{2}(1+s)\right).$$

The proof is complete. □

Remarks

- $\zeta(s)$ has simple zeros at $s = -2, -4, -6, -8, \ldots$. Indeed, since the integral in (*) is absolutely convergent for all $s \in \mathbb{C}$ and since $\omega(x) > 0$ for all $x \in \mathbb{R}$, we have
$$\Xi(-2n) = \frac{1}{2n} - \frac{1}{2n+1} + \int_1^\infty \omega(x) \left(x^{-n} + x^{n+1/2} \right) \frac{dx}{x} > 0$$
for all $n \in \mathbb{Z}_+$. The result follows from the fact that $\Gamma(s/2)$ has simple poles at $s = -2n$.
- These zeros are the only ones lying in the region $\sigma < 0$. They are called trivial zeros of the Riemann zeta-function.
- For all $0 < \sigma < 1$, we have $\zeta(\sigma) \neq 0$. Indeed, for all $\sigma > 0$ we see
$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx$$
we infer that, for all $0 < \sigma < 1$, we get
$$\left| \zeta(\sigma) - \frac{\sigma}{\sigma - 1} \right| < \sigma \int_1^\infty \frac{dx}{x^{\sigma+1}} = 1,$$
which implies that $\zeta(\sigma) < 1 + \sigma/(\sigma - 1)$ for all $0 < \sigma < 1$.
- Hence $\zeta(\sigma) < 0$ for all $\frac{1}{2} \leqslant \sigma < 1$, and the functional equation implies the asserted result.