Analytic Number Theory
Lecture 9

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Padova, April 8, 2025.

Supported by the NSF grant DMS-2154712,
and the CAREER grant DMS-2236493.

Functional equation

Theorem

Recall that $\Xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ *can be extended analytically in the whole complex plane to a meromorphic function having simple poles at* $s = 0$ *and* $s = 1$, *and satisfies the functional equation* $\Xi(s) = \Xi(1-s)$. *Let*

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$
$$= 1 + \frac{s(s-1)}{2}\int_1^\infty (\theta(x)-1)\left(x^{s/2} + x^{(1-s)/2}\right)\frac{dx}{x},$$

where θ *is the Theta function*

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}.$$

- *Then the function* $\xi(s)$ *can be extended analytically in the whole complex plane to an entire function that satisfies the functional equation* $\xi(s) = \xi(1-s)$.
- *Thus the Riemann zeta-function can be extended analytically in the whole complex plane to a meromorphic function having a simple pole at* $s = 1$ *with residue* 1. *Furthermore, for all* $s \in \mathbb{C}\backslash\{1\}$, *we have*

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

Order of ξ

Lemma

The function $\xi(s)$ *is an entire function of order* 1. *Furthermore,*

$$\limsup_{|s|\to\infty} \frac{\log|\xi(s)|}{|s|\log|s|} = \frac{1}{2}.$$

Proof.

- Since $\xi(s) = \xi(1-s)$, it suffices to bound $|\xi(s)|$ in the half-plane $\mathrm{Re}(s) \geq 1/2$.
- We can estimate $\xi(s)$ by invoking Stirling's formula, and elementary upper bounds for $\zeta(s)$. When $\mathrm{Re}(s) > 1$, we will use the identity

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx.$$

- However, since the last integral represents a holomorphic function in $\mathrm{Re}(s) > 0$, the identity holds in this larger domain.
- In particular, we have

$$|(s-1)\zeta(s)| = O(|s|^2) \quad \text{whenever} \quad \mathrm{Re}(s) \geq 1/2.$$

Proof

- Moreover, since $\log|\Gamma(s)| \leq |\log\Gamma(s)|$, Stirling's formula yields

$$\log|\Gamma(s)| \leq |s|\log|s| + O(|s|) \quad \text{whenever } \operatorname{Re}(s) \geq 1/2.$$

- Combining the last two estimates, since $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, we obtain

$$\log|\xi(s)| \leq \frac{1}{2}|s|\log|s| + O(|s|) \quad \text{whenever} \quad \operatorname{Re}(s) \geq 1/2.$$

  which establishes the first claim of the lemma.

- For the second claim, we note that

$$\log|\Gamma(|s|)| = \log\Gamma(|s|) = |s|\log|s| + O(|s|),$$

  whence

$$\log\xi(|s|) = \frac{1}{2}|s|\log|s| + O(|s|) \quad \text{as} \quad |s| \to \infty.$$

This completes the proof of the lemma. □

Zeros of function ξ

Theorem

The function $\xi(s)$ has infinitely many zeros in the strip $0 \leq \mathrm{Re}(s) \leq 1$ and no zeros outside that strip. It can be written as

$$\xi(s) = e^{Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}, \qquad (*)$$

where ρ runs through the zeros of $\xi(s)$ counted according to their multiplicities and

$$B = 1 + \frac{\gamma}{2} - \log(2\sqrt{\pi}).$$

Proof.

- By the previous lemma, we know that $\xi(s)$ has order 1.
- Noting that $\xi(0) = 1$, and using Hadamard's theorem we obtain (*) for some B.
- If $\xi(s)$ had only a finite number of zeros, (*) would imply the estimate $\log|\xi(s)| = O(|s|)$, which contradicts the fact that $\limsup_{|s|\to\infty} \frac{\log|\xi(s)|}{|s|\log|s|} = \frac{1}{2}$.
- For $\mathrm{Re}\, s > 1$, the zeta function $\zeta(s)$, and, consequently, $\xi(s)$, have no zeros in this range. By using the equation $\xi(s) = \xi(1-s)$ it follows that $\xi(s) \neq 0$ for $\mathrm{Re}\, s < 0$. Since $1 = \xi(0) = \xi(1) \neq 0$, the zeros of $\xi(s)$ lie in the strip $0 \leq \mathrm{Re}(s) \leq 1$.
- Show that $B = 1 + \frac{\gamma}{2} - \log(2\sqrt{\pi})$. □

Remarks

- The zeta function $\zeta(s)$ has simple zeros at $s = -2, -4, -6, -8, \ldots$. These zeros are called trivial zeros.
- For $\operatorname{Re} s > 1$, the zeta function $\zeta(s)$ has no zeros.
- In the critical strip $0 \leq \operatorname{Re} s \leq 1$ the zeros of

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

  are precisely the zeros of $\zeta(s)$. These zeros are called nontrivial zeros.
- We also know that $\zeta(\sigma) \neq 0$ whenever $0 < \sigma < 1$.
- In addition to the trivial zeros, the zeta function has infinitely many nontrivial zeros lying in the critical strip $0 \leq \operatorname{Re} s \leq 1$.
- The nontrivial zeros of the zeta function are distributed symmetrically with respect to the lines $\operatorname{Re} s = 1/2$ and $\operatorname{Im} s = 0$, which follows from the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

Approximate functional equation

- The functional equation is very important, but also may be insufficient in some applications for it does not express $\zeta(s)$ explicitly.
- The following tool will be useful to get some estimates of $\zeta(s)$ in the critical strip, especially when σ is close to 1.

Theorem (Exercise)

We have uniformly for $x \geqslant 1$ and $s \in \mathbb{C}\backslash\{1\}$ such that $\sigma > 0$, we have

$$\zeta(s) = \sum_{n \leqslant x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + R_0(s;x)$$

with

$$R_0(s;x) = \frac{\psi(x)}{x^s} - s\int_x^\infty \frac{\psi(u)}{u^{s+1}}du,$$

and hence

$$|R_0(s;x)| \leqslant \frac{|s|}{\sigma x^\sigma}.$$

Expansion of logarithmic derivative of ζ

Proposition

Let $s = \sigma + it$ with $-1 \leqslant \sigma \leqslant 2$ and t not equal to an ordinate of a zero of $\zeta(s)$. Set $\tau = |t| + 3$. Then we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\substack{\rho \\ |t-\mathrm{Im}\rho|\leqslant 1}} \frac{1}{s-\rho} + O(\log \tau).$$

Proof.

- Recall that $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$.
- The logarithmic differentiation provides

$$\frac{\xi'(s)}{\xi(s)} = b + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right),$$

  where the summation runs through all zeros $\rho = \beta + i\gamma$ of $\xi(s)$, which are exactly the non-trivial zeros of $\zeta(s)$.
- Moreover, the above sum is absolutely convergent, since

$$\sum_{\rho} |\rho|^{-2} < \infty.$$

Proof

- Now using the definition of $s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ gives

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} - \frac{\log \pi}{2} + \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)}.$$

- By logarithmic differentiation of $\Gamma(s)$, we obtain

$$-\frac{\Gamma'(s)}{\Gamma(s)} = \frac{1}{s} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n+s} - \frac{1}{n}\right).$$

- Therefore, we may write

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \frac{\log \pi}{2} + \frac{\gamma}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{2n+s} - \frac{1}{2n}\right) + b + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

- Note that

$$\sum_{1\le n\le \tau}\left|\frac{1}{2n+s} - \frac{1}{2n}\right| = O(\log \tau), \quad \text{and} \quad \sum_{n\ge \tau}\left|\frac{1}{2n+s} - \frac{1}{2n}\right| = O\left(\frac{|s|}{\tau}\right).$$

Proof

- Therefore, we can write that

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + O(\log \tau). \qquad (*)$$

- Notice that

$$\left|\frac{\zeta'(2+it)}{\zeta(2+it)}\right| \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2} = O(1).$$

- Applying (*) with $s = 2 + it$ and using the previous bound, we obtain

$$\left|\sum_{\rho}\left(\frac{1}{2+it-\rho} + \frac{1}{\rho}\right)\right| = O(\log \tau).$$

- Adding and subtracting the sum $\sum_{\rho}\left(\frac{1}{2+it-\rho} + \frac{1}{\rho}\right)$ from (*), we obtain

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right) + O(\log \tau).$$

Proof

- Note that

$$\mathrm{Re}\left(\frac{1}{2+it-\rho}\right) = \frac{2-\beta}{(2-\beta)^2+(t-\gamma)^2} \geq \frac{1}{4+4(t-\gamma)^2} \geq 0,$$

$$\mathrm{Re}\left(\frac{1}{\rho}\right) = \frac{\beta}{\beta^2+\gamma^2} \geq 0.$$

- Therefore, we obtain

$$\sum_{\rho} \frac{1}{4+4(t-\gamma)^2} \leq \mathrm{Re}\left(\sum_{\rho}\left(\frac{1}{2+it-\rho}+\frac{1}{\rho}\right)\right) = O(\log\tau).$$

- This immediately implies that

$$\sum_{|\mathrm{Im}\rho - t|\leq 1} 1 \leq \sum_{|\mathrm{Im}\rho - t|\leq 1} \frac{2}{1+(t-\mathrm{Im}\rho)^2} = O(\log\tau).$$

- In other words, the number of zeros ρ in the strip $t \leq \mathrm{Im}\rho \leq t+1$ is at most $O(\log\tau)$ for any $t \geq 2$.

Proof

- By the previous observation we see that

$$\sum_{\rho:|t-\gamma|\leqslant 1} \frac{1}{|2+it-\rho|} = O\Big(\sum_{\rho:|t-\gamma|\leqslant 1} 1\Big) = O(\log\tau). \qquad (**)$$

- By the previous observation we also have

$$\sum_{\rho:|t-\gamma|>1} \left|\frac{1}{2+it-\rho} + \frac{1}{\rho}\right| = O(\log\tau) \qquad (***)$$

- In order to see (***), we split $\sum_{\rho:|t-\gamma|>1} = \sum_{k\in\mathbb{Z}_+} \sum_{\rho:k<|t-\gamma|\leq k+1}$, and observe, arguing as in (**), that for each $k \in \mathbb{Z}_+$ the number of zeros ρ obeying $k < |t-\gamma| \leq k+1$ is at most $O(\log(\tau+k))$. Now let $k \in \mathbb{Z}_+$ and consider the zeros ρ satisfying $k < |\gamma - t| \leqslant k+1$. Since

$$\left|\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right| = \frac{2-\sigma}{|(s-\rho)(2+it-\rho)|} \leqslant \frac{3}{|\gamma-t|^2} \leqslant \frac{3}{k^2}$$

we infer that the contribution from the sum $\sum_{\rho:k<|t-\gamma|\leq k+1}$ is at most $O(k^{-2}\log(\tau+k))$. Summing over $k \in \mathbb{Z}_+$ we obtain (***).

Proof

- Finally, combining (**) and (***) with

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right) + O(\log\tau),$$

  we obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\substack{\rho \\ |t-\mathrm{Im}\rho|\leqslant 1}} \frac{1}{s-\rho} + O(\log\tau),$$

  as desired. □

- From the proof of the previous proposition, we obtain the following important result.

Corollary

For every real number $T \geq 2$ the number of nontrivial zeros ρ of the zeta function ζ satisfying $T \leq \mathrm{Im}\rho \leq T+1$ is at most $O(\log T)$.

Some quantitative bounds

Corollary

For every real number $T \geqslant 2$, there exists $T' \in [T, T+1]$ such that, uniformly for $-1 \leqslant \sigma \leqslant 2$, we have

$$\left| \frac{\zeta'(\sigma + iT')}{\zeta(\sigma + iT')} \right| = O(\log^2 T).$$

Proof.

- We subdivide $[T, T+1]$ into $O(\log T)$ equal parts of length $c/\log T$, where $c > 0$ is chosen so that the number of parts exceeds the number of zeros.
- By the Dirichlet pigeonhole principle, we deduce that there is a part that contains no zeros. Hence for T' lying in this part, we must have $|T' - \gamma| \geq c'/\log T$ for some $c' > 0$.
- We infer that each summand in the previous proposition is $O(\log T)$ and since there are $O(\log T)$ summands by the previous corollary, we obtain the desired estimate.

This completes the proof. □

Zero-free region estimates

Theorem (de la Vallée Poussin)

There exists an absolute constant $C > 0$ *such that* $\zeta(s)$ *has no zero* $\rho = \beta + i\gamma$ *satisfying*

$$\beta \geq 1 - \frac{C}{\log(|\gamma| + 2)}. \qquad (*)$$

Proof.

- At the point $s = 1$ the zeta function $\zeta(s)$ has a pole, and so there exists $c_1 \in \mathbb{R}_+$ so that $\zeta(s)$ has no zeros in the domain $|s - 1| \leq 2c_1$. Thus if $\rho = \beta + i\gamma$ is a nontrivial zero of $\xi(s)$ then $|\rho - 1| \geq 2c_1$. If we assume that $|\gamma| \leq c_1$, then $1 - \beta \geq c_1 \geq \frac{c_1}{4\log 2} \geq \frac{c_1}{4\log(|\gamma|+2)}$ implying (*) with $C = c_1/4$. We now fix a particular zero $\rho_0 = \beta_0 + i\gamma_0$ of $\zeta(s)$ such that $|\gamma_0| > c_1$.
- Suppose that $s = \sigma + it$ with $\sigma > 1$. Taking real parts we obtain

$$-\operatorname{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) = \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} \cos(t \log n).$$

- Since $3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0$ for any $\theta \in \mathbb{R}$, we have

$$-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4\operatorname{Re}\left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)}\right) - \operatorname{Re}\left(\frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + 2it)}\right) \geq 0. \qquad (**)$$

Proof

- Since $\zeta(s)$ has a pole of residue 1 at $s = 1$, we have

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} = \frac{1}{\sigma - 1} + O(1).$$

- We consider $s = \sigma + it$ with $t = \gamma_0$. Since $|\gamma_0| \geq c_1 > 0$, we have

$$-\operatorname{Re}\left(\frac{\zeta'(\sigma + i\gamma_0)}{\zeta(\sigma + i\gamma_0)}\right) \leq -\operatorname{Re}\sum_{|\gamma - \gamma_0| \leq 1} \frac{1}{(\sigma - \beta) + i(\gamma_0 - \gamma)} + c_2 \log(|\gamma_0| + 2)$$

$$\leq \frac{-1}{(\sigma - \beta_0)} + c_2 \log(|\gamma_0| + 2),$$

by proceeding as in the previous proposition. Similarly, we have

$$-\operatorname{Re}\left(\frac{\zeta'(\sigma + 2i\gamma_0)}{\zeta(\sigma + 2i\gamma_0)}\right) \leq c_3 \log(|\gamma_0| + 2).$$

- Inserting these three estimates into (**), we deduce that for σ close to 1,

$$4(\sigma - \beta_0)^{-1} - 3(\sigma - 1)^{-1} \leq c_4 \log(|\gamma_0| + 2)$$

- Choosing $\sigma = 1 + \frac{1}{2c_4 \log(|\gamma_0|+2)}$, we obtain

$$\beta_0 \leq 1 - \frac{1}{14 c_4 \log(|\gamma_0| + 2)},$$

which establishes (*) when $|\gamma_0| \geq c_1$. □

Important estimates

- Consider the following function

$$h(x) = \begin{cases} 1 & \text{if } x \in (1, \infty), \\ \frac{1}{2} & \text{if } x = 1, \\ 0 & \text{if } x \in (0, 1). \end{cases}$$

Lemma

Let $\kappa, T, T' \in \mathbb{R}_+$ be given.

- *If $x \neq 1$, then*

$$\left| h(x) - \frac{1}{2\pi i} \int_{\kappa - iT'}^{\kappa + iT} x^s \frac{\mathrm{d}s}{s} \right| \leq \frac{x^\kappa}{2\pi |\log x|} \left(\frac{1}{T} + \frac{1}{T'} \right).$$

- *If $x = 1$, then*

$$\left| h(1) - \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} \frac{\mathrm{d}s}{s} \right| \leq \frac{\kappa}{T + \kappa}.$$

Proof

- Consider first the case when $x > 1$. Let k be a sufficiently large integer and let $\mathcal{R}_k$ denote the rectangle with vertices $\kappa - iT'$, $\kappa + iT$, $\kappa - k + iT$, $\kappa - k - iT'$.
- Since 0 belongs to the interior of $\mathcal{R}_k$. By the Cauchy theorem, we may write
$$\frac{1}{2\pi i}\int_{\mathcal{R}_k} x^s \frac{ds}{s} = 1 = h(x).$$
- Now we have the following upper bounds

$$\left|\int_{\kappa+iT}^{\kappa-k+iT} x^s s^{-1} ds\right| \leq \int_{\kappa-k}^{\kappa} \frac{x^u du}{(u^2+T^2)^{1/2}} \leq \frac{x^\kappa}{T|\log x|},$$
$$\left|\int_{\kappa-k-iT'}^{\kappa-iT'} x^s s^{-1} ds\right| \leq \int_{\kappa-k}^{\kappa} \frac{x^u du}{(u^2+(T')^2)^{1/2}} \leq \frac{x^\kappa}{T'|\log x|},$$
$$\left|\int_{\kappa-k+iT}^{\kappa-k-iT'} x^s s^{-1} ds\right| \leq \frac{x^{\kappa-k}}{k-\kappa}\left(T+T'\right).$$

- We deduce the stated result by letting k tend to infinity.

Proof

- The case $0 < x < 1$ can be dealt with in a symmetric way, applying the same argument with k replaced by $-k$. We omit the details.
- When $x = 1$, we simply note that

$$\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} s^{-1}ds = \frac{1}{2\pi}(\arg(\kappa+iT)-\arg(\kappa-iT)) = \frac{1}{\pi}\arctan(T/\kappa).$$

- The stated upper bound is now immediate from the following bounds, valid for all $y > 0$,

$$0 \leq \frac{\pi}{2} - \arctan y = \int_y^\infty \frac{dt}{1+t^2} \leq \frac{2}{1+y}$$

This concludes the proof of the lemma. □

Perron truncated formula

- Let

$$F(s) := \sum_{n=1}^{\infty} a_n n^{-s},$$

be a Dirichlet series with abscissa of convergence σ_c and abscissa of absolute convergence σ_a.

- Its truncated variant is denoted by

$$F_{\leq x}(s) := \sum_{n \in [x]} a_n n^{-s}, \quad \text{for} \quad x \in \mathbb{R}_+.$$

Theorem (First effective Perron formula)

For $\kappa > \max\{0, \sigma_a\}$, $T \geq 1$ and $x \geq 1$, we have

$$F_{\leq x}(s) = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} F(s) x^s \frac{\mathrm{d}s}{s} + O\left(x^{\kappa} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\kappa}(1 + T|\log(x/n)|)} \right).$$

Proof

- It suffices to show that, for any fixed $\kappa > 0$, and uniformly for $y > 0, T > 0$, we have that

$$\left| h(y) - \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} y^s s^{-1} ds \right| = O\big(y^\kappa / (1 + T|\log y|)\big). \qquad (*)$$

- Indeed, applying this estimate with $y = x/n$ and summing over $n \in \mathbb{Z}_+$ (after multiplication by a_n) we obtain precisely the stated formula.
- When $T|\log y| > 1$, the estimate (*) follows from the first inequality of the previous lemma. Otherwise, we can write

$$\int_{\kappa - iT}^{\kappa + iT} y^s s^{-1} ds = y^\kappa \int_{\kappa - iT}^{\kappa + iT} s^{-1} ds + y^\kappa \int_{\kappa - iT}^{\kappa + iT} \left(y^{it} - 1\right) s^{-1} ds.$$

- The second integral is

$$O\left(\int_{-T}^{T} |(t \log y) s^{-1}| dt \right) = O(T|\log y|) = O(1),$$

and consequently, by the second inequality of the previous lemma, we see that the left-hand side of (*) is $O(y^\kappa)$. This concludes the proof. □

Perron truncated formula

Theorem (Second effective Perron formula)

Let $F(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with finite abscissa of absolute convergence σ_a.

(i) *Suppose that there exists some real number $\alpha \geq 0$ such that*

$$\sum_{n=1}^{\infty} |a_n| \, n^{-\theta} = O\big((\theta - \sigma_a)^{-\alpha} \big) \quad \textit{for} \quad \theta > \sigma_a.$$

(ii) *Assume that that B is a non-decreasing function satisfying*

$$|a_n| \leq B(n) \quad \textit{for all} \quad n \in \mathbb{Z}_+.$$

Then for $x \geq 2, T \geq 2, \sigma \leq \sigma_a$, and $\kappa := \sigma_a - \sigma + 1/\log x$, we have

$$\sum_{n \leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} F(s+w) x^w \frac{dw}{w} + O\left(x^{\sigma_a - \sigma} \frac{(\log x)^{\alpha}}{T} + \frac{B(2x)}{x^{\sigma}} \left(1 + \frac{x \log x}{T} \right) \right).$$

Proof

- Apply formula (*) to the series $\sum_{n=1}^{\infty} b_n n^{-w}$ with $b_n := a_n n^{-s}$.
- The contribution of integers $n \in \mathbb{Z}_+$ which do not belong to $\left[\frac{1}{2}x, 2x\right]$ is

$$\sum_{n \notin [x/2, 2x]} \frac{|x^\kappa||a_n|}{n^\kappa (1 + T|\log(x/n)|)} = O\left(x^\kappa T^{-1} \sum_{n=1}^{\infty} |a_n| \, n^{-\kappa-\sigma}\right)$$
$$= O\left(x^{\sigma_a - \sigma} T^{-1} (\log x)^\alpha\right).$$

- When $\frac{1}{2}x \leq n \leq 2x$, using inequality $\log y \geq 1 - \frac{1}{y}$ for $y \in \mathbb{R}_+$ we have $|\log(x/n)| \geq \frac{|x-n|}{2x}$. This leads to the following estimate

$$x^{-\sigma} \sum_{x/2 \leq n \leq 2x} \frac{|a_n|}{1 + T|\log(x/n)|} = O\left(\frac{B(2x)}{x^\sigma} \sum_{x/2 \leq n \leq 2x} \min\left\{1, \frac{x}{T|x-n|}\right\}\right).$$

- Splitting $\sum_{x/2 \leq n \leq 2x} = \sum_{x/2 \leq n \leq x-1} + \sum_{x-1 < n < x+1} + \sum_{x+1 \leq n \leq 2x}$ we obtain

$$O\left(\frac{B(2x)}{x^\sigma} \sum_{x/2 \leq n \leq 2x} \min\left\{1, \frac{x}{T|x-n|}\right\}\right) = O\left(\frac{B(2x)}{x^\sigma}\left(1 + \frac{x \log x}{T}\right)\right).$$

- This completes the proof. □

Landau's explicit formula for $\psi(x)$.

Theorem (Landau)

For any $2 \le T \le x$, we have

$$\psi(x) = x - \sum_{|\mathrm{Im}\rho| \leqslant T} \frac{x^\rho}{\rho} - \log 2\pi + O\left(\frac{x(\log x)^2}{T}\right).$$

Proof.

- We may suppose that $x \notin \mathbb{Z}$.
- For every real number $T \geqslant 2$, there exists $T' \in [T, T+1]$ such that, uniformly for $-1 \leqslant \sigma \leqslant 2$, we have

$$\left|\frac{\zeta'(\sigma + iT')}{\zeta(\sigma + iT')}\right| = O(\log^2 T).$$

- Let T' be the number supplied by the above item. Let $\mathcal{R}$ be the rectangle with vertices

$$\kappa - iT', \quad \kappa + iT', \quad -1/2 + iT', \quad \text{and} \quad -1/2 - iT'.$$

Proof

- We know that

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + B - \sum_{n=1}^{\infty}\left(\frac{1}{2n+s} - \frac{1}{2n}\right) - \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

- This implies that $-\zeta'(s)/\zeta(s)$ has simple poles at $s = -2k$ for $k \in \mathbb{Z}_+$ with residue -1, and the residue at $s = 1$, which is equal to 1.
- Therefore, by the residue theorem we obtain

$$\frac{1}{2\pi i}\int_{\mathcal{R}} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,\mathrm{d}s = x - \sum_{|\gamma|\leqslant T'}\frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)},$$

since

$$\operatorname{res}_0\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right) = -\frac{\zeta'(0)}{\zeta(0)}, \quad \text{and} \quad \operatorname{res}_1\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right) = x,$$
$$\text{and} \quad \operatorname{res}_\rho\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right) = \frac{x^\rho}{\rho}.$$

- It can be shown that $\zeta'(0)/\zeta(0) = \log 2\pi$.

Proof

- Note that $\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{n \leq x} \frac{\Lambda(n)}{n^s}$ with $s = 0$. We know that

$$\left| -\frac{\zeta'(1+\sigma+it)}{\zeta(1+\sigma+it)} \right| = -\frac{\zeta'(1+\sigma)}{\zeta(1+\sigma)} < \frac{1}{\sigma}$$

  for any $\sigma > 0$.
- Using the second Perron formula with $\sigma_a = \alpha = 1$, $\sigma = 0 = \mathrm{Re}\, s$, $\kappa = 1 + 1/\log x$ and $a_n = \Lambda(n)$ and $B(n) = \log n$ to obtain

$$\begin{aligned}\psi(x) =& x - \sum_{|\gamma| \leqslant T'} \frac{x^\rho}{\rho} - \log 2\pi - \sum_{j=1}^{2} I_{\mathcal{H}_j} - I_{\mathcal{V}} \\ &+ O\left(\frac{x(\log x)^2}{T'} + \log x \right).\end{aligned}$$

  where $I_{\mathcal{H}_j}$ denotes the integrals taken over the two horizontal sides and $I_{\mathcal{V}}$ is the integral taken over the vertical side.

Proof

- Since $T' \simeq T$, we obtain

$$I_{\mathcal{H}_j} = O\left(\int_{-1/2}^{\kappa} \left| -\frac{\zeta'(\sigma \pm iT')}{\zeta(\sigma \pm iT')} \right| \frac{x^\sigma}{|\sigma \pm iT'|} d\sigma \right)$$
$$= O\left((\log T)^2 \int_{-1/2}^{\kappa} \frac{x^\sigma}{|\sigma \pm iT'|} d\sigma \right)$$
$$= O\left(\frac{(\log T)^2}{T} \int_{-1/2}^{\kappa} x^\sigma d\sigma \right) = O\left(\frac{x(\log T)^2}{T} \right), \quad \text{for} \quad j \in [2].$$

- Moreover, we have

$$I_{\mathcal{V}} = O\left(\int_{-T'}^{T'} \left| -\frac{\zeta'(-1/2 + it)}{\zeta(-1/2 + it)} \right| \frac{x^{-1/2}}{|-1/2 + it|} dt \right)$$
$$= O\left(x^{-1/2} \int_{-T'}^{T'} \log(2 + |t|) \frac{dt}{|-1/2 + it|} \right)$$
$$= O(x^{-1/2}(\log T)^2) = O\left(\frac{x(\log T)^2}{T} \right),$$

since $2 \leq T \leq x$. This complete the proof. □

The prime number theorem (PNT)

Theorem (PNT)

There exists an absolute constant $c \in (0,1)$ such that as $x \to \infty$, one has

$$\psi(x) = x + O\big(xe^{-c\sqrt{\log x}}\big), \qquad (*)$$

$$\pi(x) = \mathrm{Li}(x) + O\big(xe^{-c\sqrt{\log x}}\big), \qquad (**)$$

where

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

Moreover, one has

$$\mathrm{Li}(x) = \frac{x}{\log x} + x\sum_{k=1}^{N-1} \frac{k!}{(\log x)^{k+1}} + O\left(\frac{x}{(\log x)^{N+1}}\right).$$

For instance, for $x \to \infty$, the estimate

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$$

is useful in many applications.

Proof

- For any fixed $N \in \mathbb{Z}_+$, by repeated integration by parts, we may derive

$$\mathrm{Li}(x) = \frac{x}{\log x} + x \sum_{k=1}^{N-1} \frac{k!}{(\log x)^{k+1}} + O\left(\frac{x}{(\log x)^{N+1}}\right).$$

- The second part (**) follows from the first part (*) by summation by parts. Therefore, it suffices to prove the first part (*).
- By Landau's theorem we obtain, for any $2 \leq T \leq x$, that

$$|\psi(x) - x| \leq \sum_{|\mathrm{Im}\rho| \leqslant T} \frac{x^{\mathrm{Re}\rho}}{|\rho|} + O\left(\frac{x(\log x)^2}{T}\right).$$

- By the zero-free region estimates, there exists an absolute constant $C \in \mathbb{R}_+$ such that for every nontrivial zero $\rho = \beta + i\gamma$ of $\zeta(s)$, we have

$$\mathrm{Re}\rho = \beta \leq 1 - \frac{C}{\log(|\gamma| + 2)} \leq 1 - \frac{C}{\log T}.$$

- Hence, inserting this bound into the previous one, we otain

$$\sum_{|\mathrm{Im}\rho| \leqslant T} \frac{x^{\mathrm{Re}\rho}}{|\rho|} \leq x^{1 - \frac{C}{\log T}} (\log T)^2.$$

- Taking $T = e^{\sqrt{\log x}}$, we obtain the desired result and (*) follows. □

Riemann hypothesis

Riemann hypothesis (1859)

All non-trivial zeros of $\zeta(s)$ are on the critical line $\mathrm{Re} s = \frac{1}{2}$.

Remark

- If the Riemann hypothesis is true, then we would have

$$\psi(x) = x + O\big(\sqrt{x}(\log x)^2\big). \qquad (*)$$

- This follows immediately by adapting the argument from the previous slide with $T = \sqrt{x}$, where the essential input comes from the fact that all nontrivial zeros ρ of the $\zeta(s)$ satisfy $\mathrm{Re}\rho = 1/2$.
- A striking result is that the asymptotic (*) implies the Riemann hypothesis.

Exercise

Prove that the Riemann hypothesis is equivalent to the PNT as in (*).

Korobov and Vinogradov's theorem

Theorem

- *For all* $s = \sigma + it \in \mathbb{C}$ *such that* $\frac{1}{2} \leqslant \sigma \leqslant 1$ *and* $t \geqslant 3$*, one has*

$$|\zeta(s)| \leqslant A t^{B(1-\sigma)^{3/2}} (\log t)^{2/3}. \qquad (*)$$

- *Inequality (*) implies that there exists an absolute constant* $c_0 > 0$ *such that* $\zeta(s)$ *has no zero in the region*

$$\sigma \geqslant 1 - \frac{c_0}{(\log |t|)^{2/3} (\log\log |t|)^{1/3}} \quad \text{and} \quad |t| \geqslant 3. \qquad (**)$$

- The estimate (**) is essentially the best zero-free region for $\zeta(s)$ up to now, which was independently obtained by Korobov and Vinogradov.

Theorem (PNT with the best error term to date)

There exists an absolute constant $c \in (0, 1)$ *such that as* $x \to \infty$*, one has*

$$\psi(x) = x + O\left(x \exp\left(-c(\log x)^{3/5} (\log\log x)^{-1/5}\right)\right),$$
$$\pi(x) = \mathrm{Li}(x) + O\left(x \exp\left(-c(\log x)^{3/5} (\log\log x)^{-1/5}\right)\right).$$

Theorem of Landau

- In fact, any order of magnitude of $\zeta(s)$ in a certain domain implies a zero-free region as may be seen in the next result devised by Landau.

Theorem (Landau)

Let $\theta(t)$ and $\phi(t)$ be positive functions such that $\theta(t)$ is decreasing, $\phi(t)$ is increasing and $e^{-\phi(t)} \leqslant \theta(t) \leqslant \frac{1}{2}$. Assume that $\zeta(s) = O(e^{\phi(t)})$ in the region $\sigma \geqslant 1 - \theta(t)$ and $t \geqslant 2$. Then the following assertions hold.

(i) *There exists an absolute constant $c_0 > 0$ such that $\zeta(s)$ has no zero in the region*

$$\sigma \geqslant 1 - c_0 \frac{\theta(2t+1)}{\phi(2t+1)}.$$

(ii) *In the region $\sigma \geqslant 1 - (c_0/2)\,\theta(2t+2)\phi(2t+2)^{-1}$, we have*

$$\frac{1}{\zeta(s)} = O\left(\frac{\theta(2t+2)}{\varphi(2t+2)}\right) \quad \textit{and} \quad \frac{\zeta'(s)}{\zeta(s)} = O\left(\frac{\theta(2t+2)}{\varphi(2t+2)}\right).$$

Consequences of the PNT

- The PNT enables us to improve on estimates for some functions of prime numbers by proceeding as follows. Let $x \geqslant 2$ be a large real number and $f \in C^2([2,x])$. Then one can write

$$\sum_{p \leqslant x} f(p) = f(x)\pi(x) - \int_2^x f'(u)\,\mathrm{Li}(u)\mathrm{d}u - \int_2^x f'(u)(\pi(u) - \mathrm{Li}(u))\mathrm{d}u$$
$$= \int_2^x \frac{f(u)}{\log u}\,\mathrm{d}u + f(2)\,\mathrm{Li}(2) + f(x)(\pi(x) - \mathrm{Li}(x))$$
$$- \int_2^x f'(u)(\pi(u) - \mathrm{Li}(u))\mathrm{d}u.$$

- If the integral $\int_2^\infty f'(u)(\pi(u) - \mathrm{Li}(u))\mathrm{d}u$ converges, then we obtain

$$\sum_{p \leqslant x} f(p) = \int_2^x \frac{f(u)}{\log u}\,\mathrm{d}u + c_f + f(x)(\pi(x) - \mathrm{Li}(x))$$
$$+ \int_x^\infty f'(u)(\pi(u) - \mathrm{Li}(u))\mathrm{d}u$$

with

$$c_f = f(2)\,\mathrm{Li}(2) - \int_2^\infty f'(u)(\pi(u) - \mathrm{Li}(u))\mathrm{d}u$$

Improvements in Mertens' theorems

- For instance, with the latest version of the error term in the PNT we obtain

$$\sum_{p\leqslant x}\frac{1}{p} = \log\log x + B + O\left(\exp\left(-c_1(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right),$$

$$\sum_{p\leqslant x}\frac{\log p}{p} = \log x - E + O\left(\exp\left(-c_1(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right),$$

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} + O\left(\exp\left(-c_1(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right),$$

for some absolute constant $c_1 > 0$, where

$$B \approx 0.261497212\ldots$$

is the Mertens constant and

$$E \approx 1.332582275\ldots.$$

The Prime Number Theorem for Arithmetic Progressions

- Let $a, q \in \mathbb{Z}_+$ be such that $(a,q)=1$. It is customary to define the function
$$\pi(x;q,a) = \sum_{\substack{p \leqslant x \\ p \equiv a (\bmod q)}} 1.$$
- We know $\lim_{x\to\infty} \pi(x;q,a) = \infty$ and hence the question of its order of magnitude arises naturally.
- We expect the prime numbers to be well distributed in the $\varphi(q)$ reduced residue classes modulo q. Applying naively the methods that we have already developed to the function
$$\Psi(x,\chi) = \sum_{n \leqslant x} \Lambda(n)\chi(n),$$
where χ is a non-principal Dirichlet character modulo q and to its Dirichlet series $-\frac{L'}{L}(s,\chi)$, we obtain
$$\pi(x;q,a) = \frac{\mathrm{Li}(x)}{\varphi(q)} + O_q\left(xe^{-c_0(q)\sqrt{\log x}}\right)$$
for some constant $0 < c_0(q) < 1$ depending on q, the constants implied in the error term depending also on q.
- This dependence makes this result useless in practice.

The Prime Number Theorem for Arithmetic Progressions

- A great deal of effort has been made to prove some efficient estimates where the constants do not depend on the modulus.
- One of the most important results in the theory is called the Siegel-Walfisz-Page theorem or Siegel–Walfisz theorem.

Theorem (Siegel–Walfisz)

Let $a, q \in \mathbb{Z}_+$ be coprime integers.

(i) *For all $A > 0$, there exists $c_1(A) \in \mathbb{R}_+$ not depending on q such that, for all $q \leqslant (\log x)^A$, we have*

$$\pi(x; q, a) = \frac{\mathrm{Li}(x)}{\varphi(q)} + O_A\left(xe^{-c_1(A)\sqrt{\log x}}\right).$$

(ii) *For all $A > 0$ and all $q \geqslant 1$, we have*

$$\pi(x; q, a) = \frac{\mathrm{Li}(x)}{\varphi(q)} + O_A\left(\frac{x}{(\log x)^A}\right).$$

The Prime Number Theorem for Arithmetic Progressions

- Obviously, if we define the Chebyshev type functions

$$\theta(x;q,a) = \sum_{\substack{p\leqslant x \\ p\equiv a (\bmod q)}} \log p \quad \text{and} \quad \psi(x;q,a) = \sum_{\substack{n\leq x \\ n\equiv a (\bmod q)}} \Lambda(n),$$

then similar estimates hold for these functions, namely

$$\left.\begin{matrix} \theta(x;q,a) \\ \psi(x;q,a) \end{matrix}\right\} = \frac{x}{\varphi(q)} + O_A\left(xe^{-c_1(A)\sqrt{\log x}}\right),$$

and

$$\left.\begin{matrix} \theta(x;q,a) \\ \Psi(x;q,a) \end{matrix}\right\} = \frac{x}{\varphi(q)} + O_A\left(\frac{x}{(\log x)^A}\right).$$

- The proof of the Siegel–Walfisz theorem rests on an explicit formula for $\psi(x,\chi)$ similar to that of the PNT theorem, namely

$$\psi(x,\chi) = E_0(\chi)x - \sum_{|\gamma|\leqslant T} \frac{x^\rho}{\rho} + O\left(\frac{x}{T}(\log qx)^2 + x^{1/4}\log x\right),$$

where

$$E_0(\chi) = \begin{cases} 1, & \text{if } \chi = \chi_0, \\ 0, & \text{otherwise.} \end{cases}$$

The Prime Number Theorem for Arithmetic Progressions

- This in turn implies that

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + \sum_{\chi (\bmod q)} \sum_{|\gamma| \leqslant T} \frac{x^\rho}{\rho} + O\left(\frac{x}{T}(\log qx)^2 + x^{1/4}\log x\right).$$

- The other important tool is the knowledge of a zero-free region for the function $L(s,\chi)$. The arguments generalize those of the function $\zeta(s)$, except that there is an unforeseen difficulty in connection with the possible existence, still unproven, of an exceptional zero $\beta_1 \in \mathbb{R}$ near the point 1 of a function $L(s,\chi)$ attached to a quadratic Dirichlet character. More precisely, we have the following result.

Theorem (Zero-free region for L-functions)

Let $q \in \mathbb{Z}_+$ and $\tau = |t| + 3$. There exists an absolute constant $c_0 > 0$ such that if χ is a Dirichlet character modulo q, then the function $L(s,\chi)$ has no zero in the region

$$\sigma \geqslant 1 - \frac{c_0}{\log(q\tau)}$$

unless χ is a quadratic character, in which case $L(s,\chi)$ has at most one, necessarily real, zero $\beta_1 < 1$ in this region. This zero is called exceptional.

- At the present time, we do not know much more about this exceptional zero. Nevertheless, Landau, Page and Siegel provided some very important results, showing in particular that such a zero occurs at most rarely.

The Prime Number Theorem for Arithmetic Progressions

Theorem

Let $q \in \mathbb{Z}_+$ and $\tau = |t| + 3$.

(i) *(Landau). There exists an absolute constant $c_1 > 0$ such that the function $\prod_{\chi(\bmod q)} L(s, \chi)$ has at most one zero in the region*

$$\sigma \geqslant 1 - \frac{c_1}{\log(q\tau)}.$$

If such a zero β_1 exists, then it is necessarily real and associated to a quadratic character χ_1. This character is called the exceptional character.

(ii) *(Page). If χ is a Dirichlet character modulo q, then $L(\sigma, \chi) \neq 0$ in the region*

$$\sigma \geqslant 1 - \frac{c_2}{q^{1/2}(\log(q+1))^2},$$

where $c_2 > 0$ is an effectively computable absolute constant.

(iii) *(Siegel). Let χ be a quadratic Dirichlet character modulo q. For all $\varepsilon > 0$, there exists a non-effectively computable constant $c_\varepsilon > 0$ such that*

$$L(1, \chi) > \frac{c_\varepsilon}{q^\varepsilon}.$$

This implies that, if χ is a quadratic character modulo q, then $L(\sigma, \chi) \neq 0$ in the region

$$\sigma \geqslant 1 - \frac{c_\varepsilon}{q^\varepsilon}.$$